Roots of Trinomials from the Viewpoint of Amoeba Theory

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Given a Laurent polynomial $f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ the *amoeba* $\mathcal{A}(f)$ (introduced by Gelfand, Kapranov, and Zelevinsky '94) is the image of its variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ under the $\mathrm{Log}|\cdot|$-map

$$\mathrm{Log}|\cdot| : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|).$$

Amoebas have an amazing amount of structural properties and are related to various mathematical subjects including complex analysis, the topology of real algebraic curves, and crystal shapes. In particular they form a canonical connection between algebraic geometry and tropical geometry. In the first part of this talk I will give an overview about selected key theorems and current key problems in amoeba theory.

In the second part of the talk we investigate an application of amoeba theory. The behavior of norms of roots of univariate trinomials $z^{s+t} + pz^t + q \in \mathbb{C}[z]$ for fixed support $A = \{0, t, s+t\} \subset \mathbb{N}$ with respect to the choice of coefficients $p, q \in \mathbb{C}$ is a classical late 19th and early 20th century problem. Although algebraically characterized by P. Bohl in 1908, the geometry and topology of the corresponding parameter space of coefficients is unknown. We provide such a characterization for the space of trinomials T_A by reinterpreting the problem in terms of amoeba theory. The roots of given norm are parameterized in terms of hypotrochoid and epitrochoid curves along a $\mathbb{C}$-slice of the parameter space of coefficients of trinomials. Multiple roots of this norm appear exactly on the singularities of the curves. This reproves a classical statement by Sommerville from 1920.

Moreover, we show that the set of all trinomials with support A and certain roots of identical norm as well as its complements can be deformation retracted to the torus knot $K(s+t, s)$, and thus are connected but not simply connected.

The second part is based on joint work with Thorsten Theobald.