The topos-theoretic multiverse: a modal approach for computation

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Abstract. To help put established results in constructive algebra and constructive combinatorics into perspective, construct an origin story for certain inductive definitions and form a unified framework for certain techniques for extracting programs from classical proofs, we propose a modal study of the topos-theoretic multiverse. Our proposal is inspired by the corresponding study of the set-theoretic multiverse, but focuses less on exploring the range of set/topos-theoretic possibility and more on concrete applications in constructive mathematics.

Thanks to the finer distinctions constructive mathematics offers, there is a host of principles which are available in classical mathematics but seem naive from a constructive point of view. A non-exhaustive list is:

1*. A transitive relation is well-founded iff there is no infinite descending chain.
2*. A relation is almost-full iff every infinite sequence is good.[3]
3*. *Krull's lemma*: A ring element is nilpotent iff every prime ideal contains it.
4*. Every ring has a maximal ideal.[4]
5*. *Markov's principle:* If a function $\mathbb{N} \to \mathbb{N}$ does *not not* have a zero, then it actually has a zero.
6*. *Dependent choice:* If every element of a set is related by some relation to some other element, then every element can be completed to an infinite chain of related elements.
7*. The law of excluded middle holds.

Constructive theorems always carry computational and geometric content—from every constructive proof, a corresponding algorithm can be extracted [2], and every constructive proof holds also for continuous families of the objects in question [4, Section 4.3]. In contrast, the listed classical principles above have no

[3] An infinite sequence is *good* iff some term of the sequence is related to some later term. The notion of almost-full relations has been studied in combinatorics [17,7,12] and found applications in termination checking [3].

[4] Here and in the following, by *ring* we mean commutative ring with unit and by *maximal ideal* we mean an ideal $\mathfrak{m}$ which is *proper* in the sense that $1 \in \mathfrak{m} \Rightarrow 1 = 0$ and such that for every proper ideal $\mathfrak{n}$ with $\mathfrak{m} \subseteq \mathfrak{n}$, $\mathfrak{m} = \mathfrak{n}$.

computational witness and/or fail in continuous families, hence are not available in constructive mathematics.

In the modal topos-theoretic multiverse, we have the following constructive replacements to these principles.

1. A transitive relation is well-founded iff *everywhere* there is no infinite descending chain.
2. A relation is almost-full iff every infinite sequence *everywhere* is good.
3. A ring element is nilpotent iff all prime ideals *everywhere* contain it.
4. Every ring *proximally* has a maximal ideal.
5. If a function $\mathbb{N} \to \mathbb{N}$ does *everywhere not not* have a zero, then it actually has a zero.
6. If every element of a set is related by some relation to some other element, then every element can *proximally* be completed to an infinite chain of related elements.
7. *Barr's theorem, simple version: Somewhere,* the law of excluded middle holds.

Briefly, a statement φ is said to hold *everywhere* ($\Box\varphi$) iff it holds in every Grothendieck topos over the current base topos (making use of the *internal language* of toposes [6,16,13]);[5] A statement holds *somewhere* ($\Diamond\varphi$) iff it holds in some positive Grothendieck topos over the current base; and a statement holds *proximally* ($\diamondsuit\!\!\!\cdot\,\varphi$) iff it holds in some positive overt Grothendieck topos over the current base. More such *modalities* are also useful and merit study; the precise definitions are given in Section 1.

This modal language not only allows us to recover classical principles as above, but also makes some powerful theorems about the topos-theoretic landscape smoothly accessible:

8. *Barr's theorem, full version:* If Zorn's lemma holds, it is everywhere the case that it (and even the full axiom of choice) hold somewhere.
9. If a geometric sequent $\wedge_{i=1}^{n}\phi_i \vdash \bigvee_{j\in J}\psi_j$ holds *somewhere*, then it holds already here.
10. If a bounded first-order statement holds *proximally*, then it holds already here.
11. For every (perhaps uncountable) inhabited set X, *proximally* there is a surjection $\mathbb{N} \twoheadrightarrow X$.

An example application of the latter two principles has recently been studied in constructive commutative algebra [5]: For countable rings, an explicit iterative construction of a maximal ideal is available. By Item 11, this construction can also be carried out for arbitrary rings, though the result is not a maximal ideal in the narrow sense; rather, the resulting maximal ideal exists *proximally*, in some positive overt Grothendieck topos. However, the first-order consequences of its existence, pertaining for instance to concrete statements about polynomials or matrices, pass down to the base topos by Item 10. The resulting maximal ideal

[5] In order to support unbounded quantification we occasionally make use of the an extension of the usual Kripke–Joyal semantics in the form of *stack semantics* [15])

can be regarded as a mathematical phantom in the sense of Gavin Wraith [18], not existing in a narrow sense (in the base topos), but existing proximally and hence encouraging us to broaden our notion of existence because it promises us to work wonders.

We can also adopt the notions of *switches* and *buttons* from the modal study of the set-theoretic multiverse [11]. Switches are statements φ such that $\Box(\Diamond\varphi \wedge \Diamond\neg\varphi)$, while buttons are statements φ such that $\Box\Diamond\Box\varphi$; switches can be toggled on and off like a light switch, while buttons once pressed cannot be unpressed:

11. *The law of excluded middle is a switch:* Everywhere it is the case that somewhere LEM holds and somewhere it does not.
12. *Being countable is a button:* For every set X, everywhere it is the case that somewhere (even proximally) it is the case that everywhere X is countable.

We argue that the modal operators $\Box, \Diamond, \diamondsuit$ and more suggested are natural extensions and refinements of the familiar double-negation modality $\neg\neg$ in constructive mathematics.

Related work. The idea of a mathematical multiverse is not new, but arguably basic to topos theory. We aim to present a more systematic study of the modal nature of the multiverse with a focus on applications in constructive mathematics. Specific precursors to XXX

In the airy reaches of classical set theory a related philosophy is put forward, with a number of striking results, concentrating on exploring the range of set-theoretic possibility [9,8,10,14,1].

This text is set in the context of constructive mathematics. Our definitions and results can be formalized in IZF or in the kind of language supported by toposes.

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1 Modal operators

Definition 1.1. *A* Grothendieck topos *is a category equivalent to the category of sheaves on a small site. A* Grothendieck topos over a given (Grothendieck or elementary) topos $\mathcal{B}$ *is "a Grothendieck topos from the point of view of $\mathcal{B}$"; using a sufficiently expressive form of the internal language of $\mathcal{B}$, which allows us to make direct sense of the notion in scare quotes, this amounts to a bounded geometric morphism $\mathcal{E} \to \mathcal{B}$, and this notion can be taken as the definition.*

Henceworth, by the unqualified word "topos" we will mean "Grothendieck topos over the current base", and the current base topos will be denoted "Set".

The base topos might be the "true category of sets", assuming that this concept is available in one's ontology, or also some other elementary topos such as the free topos [**?**] or the effective topos [**?**]. The reader is reminded that the nature of the base topos can have profound consequences for the truths of the internal language.

The topos-theoretic multiverse of a base $\mathcal{B}$ is the collection of Grothendieck toposes over $\mathcal{B}$ (which in set-theoretic foundations should more precisely be formalized as the proper class of small sites in $\mathcal{B}$).

Remark 1.2. Perhaps arbitrary *elementary* toposes over the base, corresponding to possibly unbounded geometric morphisms $\mathcal{E} \to \mathcal{B}$, or even arbitrary fibrations/indexed categories over $\mathcal{B}$ validating an appropriate form of the axioms of elementary toposes, should also be taken as part of the topos-theoretic multiverse. The first foray into the modal topos-theoretic multiverse outlined in this note sticks to Grothendieck toposes for ease of formalizability ("for every small site" can be expressed also in the more standard flavors of the internal topos language, while "for every elementary topos" requires more elaborate versions); because the restricted multiverse is already sufficiently rich for the intended applications; and because we have a generalization to the predicative setting [**?**] with arithmetic universes in mind. Predicatively, not even the category of sets might be an elementary topos, but the category of sets and categories of sheaves are still arithmetic universes.

1.1 Positive toposes

Definition 1.3. *A topos $\mathcal{E}$ is* positive *if and only if the unique geometric morphism $f : \mathcal{E} \to \mathrm{Set}$ is surjective (that is f^* reflects isomorphisms).*

Example 1.4. The topos of sheaves over a topological space X is positive if and only if X has a point; more generally, the topos of sheaves over a locale X is positive if and only if every open covering of the top element of the frame of X is inhabited. As such, positivity is a more informative version of the constructively weaker property of being nontrivial (the property that the top open and the bottom open do not coincide).

Example 1.5. The spectrum of a ring A, that is the classifying topos of prime filters of A (or equivalently the topos of sheaves over the classifying locale of prime filters of A) is positive iff $1 \neq 0$ in A.

Example 1.6. A necessary and sufficient criterion for the classifying topos of a geometric theory $\mathbb{T}$ to be positive is that whenever $\top \vdash \bigvee_{i \in I} \varphi_i$ modulo $\mathbb{T}$, then I is inhabited.

1.2 Overt toposes

Definition 1.7. *A topos $\mathcal{E}$ is* overt *if and only if the unique geometric morphism $f : \mathcal{E} \to \mathrm{Set}$ is open (that is f^* preserves the interpretation of bounded first-order formulas).*

Example 1.8. The topos of sheaves over a topological space is always overt. More generally, the topos of sheaves over a locale X is overt iff there exists a *positivity predicate* on its frame of opens in the sense of [**?**]. With LEM, every locale and indeed every topos is overt [**?**].

Example 1.9. The spectrum of a ring A is overt iff every element of A is nilpotent or not [**?**, Proposition 12.51].

Example 1.10. A sufficient criterion for the classifying topos of a geometric theory $\mathbb{T}$ being overt is that the indexing sets of all disjunctions appearing on the right hand side of turnstiles, in a normal form presentation of $\mathbb{T}$, are inhabited [**?**, Proposition V.3.2].

1.3 Modal operators

2 Generic models and inductive definitions

In constructive mathematics, the classical definition of well-founded relations as those transitive relations for which there exist no infinite descending chains is not particularly useful; while the chain condition is satisfied for the intended examples (such as $(\mathbb{N}, <)$), by its negativity the condition is too weak to facilitate the intended proofs.

The established substitute is to declare that a transitive relation $(<)$ on a set X is well-founded if and only if for every subset $M \subseteq X$,

$$\big(\forall x : X.\ (\forall y : X.\ y < x \Rightarrow y \in M) \Rightarrow x \in M\big) \Longrightarrow \big(\forall x : X.\ x \in M\big). \qquad (\star)$$

More economically, and preferably in predicative contexts where there is no single set or class of "all subsets of X" but for instance a hierarchy of subsets of increasing universe levels, a transitive relation $(<)$ is declared well-founded if and only if every element of X is *accessible*, where the accessibility predicate Acc is inductively generated [**?**] by the following clause:

$$\frac{\forall y < x.\ \ \mathsf{Acc}(y)}{\mathsf{Acc}(x)}$$

In impredicative settings, this inductive definition of well-foundedness coincides with the higher-order characterization $(\star)$, and for the purposes of this paper we view the inductive definition as the official one.

Similar inductive notions are used to reformulate other classical definitions in a constructively more sensible way. For instance, the classical definition of a binary relation R on a set X being *almost-full* is "every infinite sequence $\alpha : \mathbb{N} \to X$ is *good* in the sense that there exist indices $i < j$ such that $\alpha(i)\ R\ \alpha(j)$". For a constructive reformulation, we shift to finite approximations of infinite sequences (finite lists of elements of X) and define when such an approximation is deemed good:[6]

$$\mathsf{Good}(\sigma) :\equiv (\exists i < j.\ \sigma[i]\ R\ \sigma[j]).$$

We then inductively generate a relation "$P \mid \sigma$" for monotonous predicates P on finite lists expressing that no matter how the given finite approximation σ evolves over time to a better approximation, eventually P will hold, by the clauses

$$\frac{P(\sigma)}{P \mid \sigma} \qquad \frac{\forall x : X.\ \ P \mid \sigma x}{P \mid \sigma}$$

and finally define that R is almost-full iff "$\mathsf{Good} \mid []$". With this inductive definition, expected properties of the class of almost-full relations such as stability under cartesian products (Dickman's lemma), finite lists (Higman's lemma) or finitely-branching trees (Kruskal's theorem) can all be constructively verified [**?**].

A similar such definition has been proposed by Thierry Coquand, Henry Lombardi and Henrik Persson in commutative algebra for expressing that a ring is Noetherian [**?**,**?**,**?**]; the classical definition "every ascending chain of ideals stabilizes" and also the more meaningful and classically equivalent characterization as "every ascending chain of finitely generated ideals stalls"[7] are constructively too weak; firstly, without the axiom of dependent choice we can often not construct such chains [**?**] (but only "multi-valued chains" as in [**?**, Section 3.9]; but also see [**?**, Section 4]), and secondly, being able to inspect suitable inductive witnesses enables us to prove the Hilbert basis theorem [**?**, Corollary 16]. Coquand, Lombardi and Persson hence propose to call a ring *Noetherian* if and only if $\mathsf{Stalls} \mid [\,]$, where Stalls is the predicate on finite lists of finitely generated ideals expressing xxx.

Is there a deeper explanation where these inductive definitions come from, apart from working well and being motivated on general constructive considerations? Also: Constructively the inductive definitions are much stronger than their classical counterparts, equivalent only in presence of LEM and DC. For instance, if a relation is almost-full in the inductive sense, not only is every infinite sequence good, but so is every infinite "multi-valued sequence"[8] and every infinite partially-defined sequence α for which for every number $n \in \mathbb{N}$ it is *not not* the case that $\alpha(n)$ exists. Can we pinpoint how much stronger the inductive definitions are?

Both questions have positive answers, and the modal perspective fruitfully clarifies their connection.

Namely, the theories of an infinite sequence and of an infinite descending chain are geometric. As such, there exist their classifying toposes, containing

[6] By "$\sigma[n]$", we mean the element at position n of the finite list σ. This notation is only meaningful if the length of σ is at least $n + 1$. By "σx" we mean the enlarged list which has x as an extra element at its tail end, and by "$[\,]$" we denote the empty list. In computer science practice, it is often more efficient to prepend ("$x\sigma$") instead of append, but this detail shall not concern us here.

[7] A chain $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots$ *stalls* iff for some index $n \in \mathbb{N}$, $\mathfrak{a}_{n+1} = \mathfrak{a}_n$. We are grateful to Matthias Hutzler for proposing this terminology.

[8] xxx

the *generic infinite sequence* respectively the *generic infinite descending chain*, and we may ask: When is this sequence good respectively when does this chain validate $\bot$?

Proposition 2.1. *Let R be a relation on a set X. The generic infinite sequence over X is good if and only if R is almost-full in the inductive sense.*

Proof. The classifying topos of the theory of an infinite sequence over X can be presented as the topos of sheaves over the site given by the partially ordered set of finite lists of elements of X with coverage given by xxx (see Appendix xxx) [**?**, Example 4.3], [**?**, xxx]. The Kripke–Joyal semantics states that the statement "α_0 is good", where α_0 is the generic infinite sequence, holds in the classifying topos if and only if there is a covering $\mathcal{U}$ of [] such that for every open $U \in \mathcal{U}$, xxx. This precisely amounts to R being almost-full in the inductive sense.

Corollary 2.2. *Let R be a relation on a set X. Then R is almost-full in the inductive sense if and only if everywhere, every infinite sequence is good.*

Proof. For the "if" direction, if every infinite sequence everywhere is good, then in particular the generic infinite sequence is. By Proposition 2.1, this statement amounts to R being almost-full in the inductive sense.

In the converse direction, we can either argue that, since being good is expressible as a geometric formula, if the generic infinite sequence is good then so is every infinite sequence in every topos; or we argue, using Proposition 2.1 again, that the property of being almost-full in the inductive sense is stable under pullback along geometric morphisms and hence passes from the base topos to every topos. Hence (the pullback of) R is almost-full in every topos and hence every infinite sequence in every topos is good.

Proposition 2.3. *Let $(<)$ be a transitive relation on a set X. The generic infinite descending chain over X validates $\bot$ (that is, the classifying topos of such chains is trivial) if and only if the relation is well-founded in the inductive sense.*

Proof. Similar as the proof of Proposition 2.1. Details for the variant of "bad sets" instead of "infinite descending chains" have been developed (xxx:language) by Blass [**?**].

Corollary 2.4. *A transitive relation is well-founded in the inductive sense if and only if everywhere, it is not the case that there exists an infinite descending chain.*

Proof. Analogous to the proof of Corollary 2.2.

3 Extracting programs from multiverse proofs

Proposition 3.1. *Let $(\leq)$ be a transitive almost-full relation. Then $(<)$, where $x < y \equiv (x \leq y \wedge \neg(y \leq x))$, is well-founded.*

Proof. Everywhere, there can be no infinite descending chain, as any such would also be good.

Unrolling this proof gives a program of type $(\mathsf{Good} \mid []) \to \prod_{x:X} \mathsf{Acc}(x)$.

Remark 3.2. For the proof of Proposition 3.1, it is not relevant that pullback along geometric morphisms typically fails to preserve the negation occuring in the definition of $(<)$, basically because we still have $f^*([\![x > y]\!]) \wedge f^*([\![x \leq y]\!]) \Rightarrow f^*([\![\bot]\!]) = \bot$.

Theorem 3.3 (Dickson's lemma). *If X and Y are almost-full, so is $X \times Y$.*

Proof.
1. It suffices to verify that the *generic infinite sequence* $\gamma = (\alpha, \beta) : \mathbb{N} \to X \times Y$ is good. Since being good can be put as a geometric implication (in fact, a geometric formula) and since LEM holds *somewhere*, we may assume LEM.
2. The set $I := \{n \in \mathbb{N} \mid \neg \exists m > n.\ \alpha(n) \leq \alpha(m)\}$ is not in bijection with $\mathbb{N}$, as else the I-extracted subsequence of α would be an X-sequence which is not good. Hence, by LEM, the set I is finite. Every index larger than all the indices in I is a suitable starting point for an infinite ascending chain $\alpha(i_0) \leq \alpha(i_1) \leq \ldots$.[9]
3. Because Y is almost-full, the sequence $\beta(i_0), \beta(i_1), \ldots$ is good, that is there exists a pair of indices $n < m$ such that $\beta(i_n) \leq \beta(i_m)$. As also $\alpha(i_n) \leq \alpha(i_m)$, the sequence γ is good.

4 Perspectives

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[9] We can avoid dependent choice here by always picking the least possible next index.

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