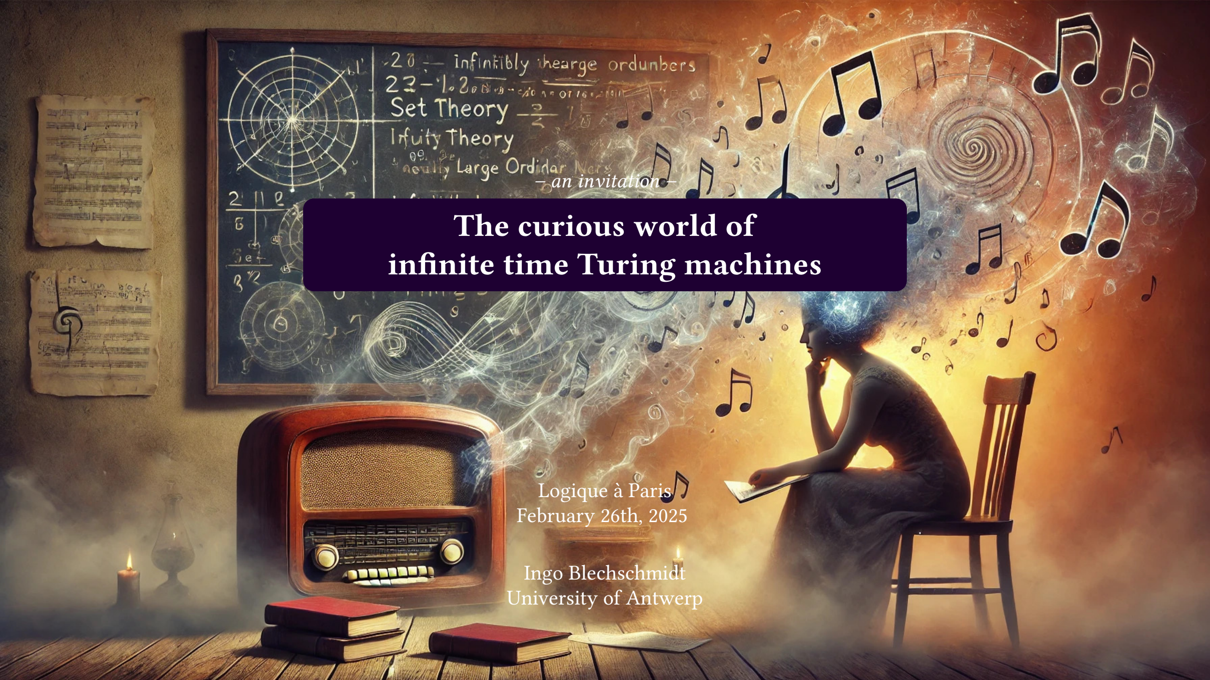


– an invitation –

The curious world of infinite time Turing machines

Logique à Paris
February 26th, 2025

Ingo Blechschmidt
University of Antwerp

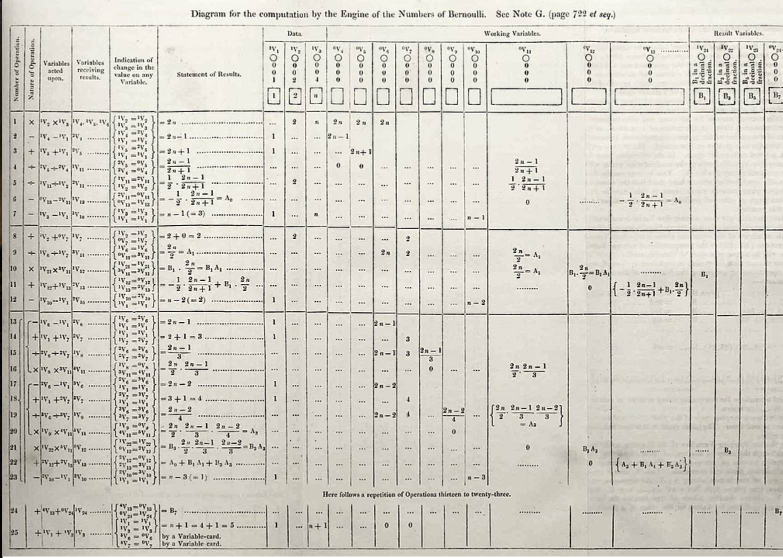


Diagram for the computation by the Engine of the Numbers of Bernoulli. See Note G. (page 722 *et seq.*)

Number of Operation.	Nature of Operation.	Variables acted upon.	Variables receiving results.	Indication of change in the value on any Variable.	Statement of Results.
1	×	${}^1V_2 \times {}^1V_3$	${}^1V_4, {}^1V_5, {}^1V_6$	$\left\{ {}^1V_2 = {}^1V_2;\ {}^1V_3 = {}^1V_3 \right\}$	$= 2n$
2	−	${}^1V_4 - {}^1V_1$	${}^2V_4$	$\left\{ {}^1V_4 = {}^2V_4;\ {}^1V_1 = {}^1V_1 \right\}$	$= 2n - 1$
3	+	${}^1V_5 + {}^1V_1$	${}^2V_5$	$\left\{ {}^1V_5 = {}^2V_5;\ {}^1V_1 = {}^1V_1 \right\}$	$= 2n + 1$
4	÷	${}^2V_5 \div {}^2V_4$	${}^1V_{11}$	$\left\{ {}^2V_5 = {}^0V_5;\ {}^2V_4 = {}^0V_4 \right\}$	$= \frac{2n-1}{2n+1}$
5	÷	${}^1V_{11} \div {}^1V_2$	${}^2V_{11}$	$\left\{ {}^1V_{11} = {}^2V_{11};\ {}^1V_2 = {}^1V_2 \right\}$	$= \frac{1}{2} \cdot \frac{2n-1}{2n+1}$
6	−	${}^0V_{13} - {}^2V_{11}$	${}^1V_{13}$	$\left\{ {}^2V_{11} = {}^0V_{11};\ {}^0V_{13} = {}^1V_{13} \right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$
7	−	${}^1V_3 - {}^1V_1$	${}^1V_{10}$	$\left\{ {}^1V_3 = {}^1V_3;\ {}^1V_1 = {}^1V_1 \right\}$	$= n - 1 (= 3)$
8	+	${}^1V_2 + {}^0V_7$	${}^1V_7$	$\left\{ {}^1V_2 = {}^1V_2;\ {}^0V_7 = {}^1V_7 \right\}$	$= 2 + 0 = 2$
9	÷	${}^1V_6 \div {}^1V_7$	${}^3V_{11}$	$\left\{ {}^1V_6 = {}^1V_6;\ {}^0V_{11} = {}^3V_{11} \right\}$	$= \frac{2n}{2} = A_1$
10	×	${}^1V_{21} \times {}^3V_{11}$	${}^1V_{12}$	$\left\{ {}^1V_{21} = {}^1V_{21};\ {}^3V_{11} = {}^3V_{11} \right\}$	$= B_1 \cdot \frac{2n}{2} = B_1 A_1$
11	+	${}^1V_{12} + {}^1V_{13}$	${}^2V_{13}$	$\left\{ {}^1V_{12} = {}^0V_{12};\ {}^1V_{13} = {}^2V_{13} \right\}$	$= -\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$
12	−	${}^1V_{10} - {}^1V_1$	${}^2V_{10}$	$\left\{ {}^1V_{10} = {}^2V_{10};\ {}^1V_1 = {}^1V_1 \right\}$	$= n - 2 (= 2)$
13	−	${}^1V_6 - {}^1V_1$	${}^2V_6$	$\left\{ {}^1V_6 = {}^2V_6;\ {}^1V_1 = {}^1V_1 \right\}$	$= 2n - 1$
14	+	${}^1V_1 + {}^1V_7$	${}^2V_7$	$\left\{ {}^1V_1 = {}^1V_1;\ {}^1V_7 = {}^2V_7 \right\}$	$= 2 + 1 = 3$
15	÷	${}^2V_6 \div {}^2V_7$	${}^1V_8$	$\left\{ {}^2V_6 = {}^2V_6;\ {}^2V_7 = {}^2V_7 \right\}$	$= \frac{2n-1}{3}$
16	×	${}^1V_8 \times {}^3V_{11}$	${}^4V_{11}$	$\left\{ {}^1V_8 = {}^0V_8;\ {}^3V_{11} = {}^4V_{11} \right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3}$
17	−	${}^2V_6 - {}^1V_1$	${}^3V_6$	$\left\{ {}^2V_6 = {}^3V_6;\ {}^1V_1 = {}^1V_1 \right\}$	$= 2n - 2$
18	+	${}^1V_1 + {}^2V_7$	${}^3V_7$	$\left\{ {}^2V_7 = {}^3V_7;\ {}^1V_1 = {}^1V_1 \right\}$	$= 3 + 1 = 4$
19	÷	${}^3V_6 \div {}^3V_7$	${}^1V_9$	$\left\{ {}^3V_6 = {}^3V_6;\ {}^3V_7 = {}^3V_7 \right\}$	$= \frac{2n-2}{4}$
20	×	${}^1V_9 \times {}^4V_{11}$	${}^5V_{11}$	$\left\{ {}^1V_9 = {}^0V_9;\ {}^4V_{11} = {}^5V_{11} \right\}$	$= \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{4} = A_3$
21	×	${}^1V_{22} \times {}^5V_{11}$	${}^0V_{12}$	$\left\{ {}^1V_{22} = {}^1V_{22};\ {}^0V_{12} = {}^2V_{12} \right\}$	$= B_3 \cdot \frac{2n}{2} \cdot \frac{2n-1}{3} \cdot \frac{2n-2}{3} = B_3 A_3$
22	+	${}^2V_{12} + {}^2V_{13}$	${}^3V_{13}$	$\left\{ {}^2V_{12} = {}^0V_{12};\ {}^2V_{13} = {}^3V_{13} \right\}$	$= A_0 + B_1 A_1 + B_3 A_3$
23	−	${}^2V_{10} - {}^1V_1$	${}^3V_{10}$	$\left\{ {}^2V_{10} = {}^3V_{10};\ {}^1V_1 = {}^1V_1 \right\}$	$= n - 3 (= 1)$
					Here follows a repetition of Operations thirteen to twenty-three.
24	+	${}^4V_{13} + {}^0V_{24}$	${}^1V_{24}$	$\left\{ {}^4V_{13} = {}^0V_{13};\ {}^0V_{24} = {}^1V_{24} \right\}$	$= B_7$
25	+	${}^1V_1 + {}^1V_3$	${}^1V_3$	$\left\{ {}^1V_1 = {}^1V_1;\ {}^1V_3 = {}^1V_3;\ {}^5V_6 = {}^0V_6;\ {}^5V_7 = {}^0V_7 \right\}$	$= n + 1 = 4 + 1 = 5$; by a Variable-card.; by a Variable card.

Ada Lovelace (* 1815, † 1852),

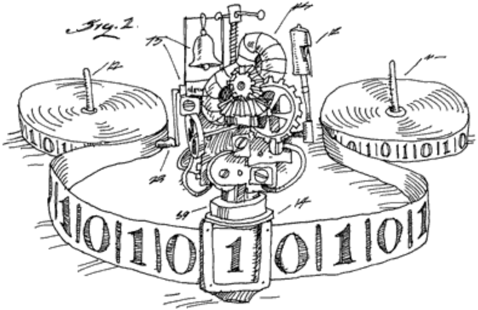
the world's first computer programmer

Part I

A crash course on ordinal numbers

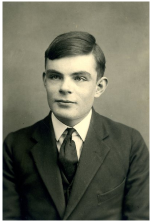
Part II

(Infinite time) Turing machines

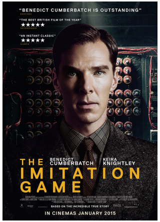


Basics on Turing machines

- Turing machines are idealized computers operating on an **infinite tape** according to a **finite list** of rules.
- The concept is astoundingly robust.
- A subset of $\mathbb{N}$ is **enumerable by a Turing machine** if and only if it is a Σ_1-set.



Alan Turing
(* 1912, † 1954)



worth watching

Alison Bechdel
(* 1960)

Infinite time Turing machines

With infinite time Turing machines, the time axis is more interesting:

- ordinary: $0,\ 1,\ 2,\ \ldots$
- infinite time: $0,\ 1,\ 2,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \omega\cdot 2,\ \omega\cdot 2+1,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

On reaching a **limit ordinal** time step like ω or $\omega \cdot 2, \ldots$

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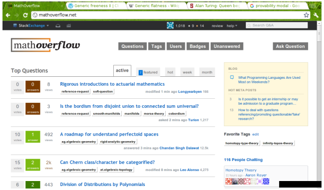
- ordinary: $0,\ 1,\ 2,\ \ldots$
- infinite time: $0,\ 1,\ 2,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \omega\cdot 2,\ \omega\cdot 2+1,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

On reaching a **limit ordinal** time step like ω or $\omega \cdot 2, \ldots$

- the machine is put into a **designated state**,
- the read/write head is **moved to the start** of the tape, and
- the tape is set to the **"lim sup"** of all its previous contents.

Joel David Hamkins



MathOverflow

Andy Lewis

A question for you

What is the behavior of this infinite time Turing machine?

In the start state and the limit state, check whether the current cell contains a “1”.

- If yes, then stop.
- If not, then flash that cell: set it to “1”, then reset it to “0”. Then unremittingly move the head rightwards.

A question for you

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In the start state and the limit state, check whether the current cell contains a “1”.

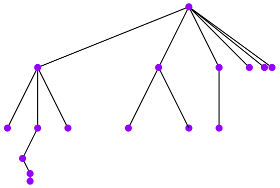
- If yes, then stop.
- If not, then flash that cell: set it to “1”, then reset it to “0”. Then unremittingly move the head rightwards.

This machine halts after time step ω^2.

Infinite time Turing machines can break out of (some kinds of) infinite loops.

What can infinite time Turing machines do?

- Everything ordinary Turing machines can do.
- Verify number-theoretic statements.
- Decide whether a given ordinary Turing machine halts.
- Simulate infinite time Turing machines.
- Decide Π^1_1- and Σ^1_1-statements:
  - “For every function $\mathbb{N} \to \mathbb{N}$ it holds that …”
  - “There is a function $\mathbb{N} \to \mathbb{N}$ such that …”



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  - "For every function $\mathbb{N} \to \mathbb{N}$ it holds that …"
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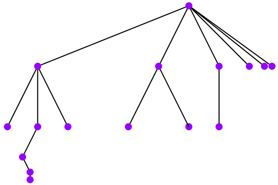
But: Infinite time Turing machines cannot compute all functions and cannot write arbitrary 0/1-sequences to the tape.



Fun facts

- Every infinite time Turing machine either halts or gets caught in an unbreakable infinite loop after **countably many steps**.
- An ordinal number α is **clockable** iff there is an infinite time Turing machine which halts precisely after time step α.
    - Speed-up Lemma: If $\alpha + n$ is clockable, then so is α.
    - Big Gaps Theorem
    - Many Gaps Theorem
    - Gapless Blocks Theorem
- **Lost Melody Theorem:** There are 0/1-sequences which a infinite time Turing machine can recognize, but not write to the tape.
- Infinite time Turing machines can be **simulated by ordinary differential equations** [Olivier Bournez, Sabrina Ouazzani]

Part III

The effective topos

The effective topos

statement	in Set	in Eff(TM)	in Eff(ITTM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 Beyond every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ 1" amounts to: There is a Turing machine which determines of any given number whether it is prime or not.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ 2" amounts to: There is a Turing machine which, given a number n, computes a prime larger than n.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"$\mathrm{Eff}(\mathrm{TM}) \models$ 3" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, determines whether f has a zero or not.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) $\models$ 4" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs a Turing machine computing f.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(ITTM) $\models$ 4" amounts to: There is an infinite time Turing machine which, given an *infinite time* Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs an (*ordinary*) Turing machine computing f.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	?
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

A real number of Eff(TM) is externally represented by a Turing machine M which on input n outputs a rational approximation $M(n)$. These approximations need to be compatible in that $|M(n) - M(m)| \leq 2^{-n} + 2^{-m}$ for all n, m.

Two such machines M and M' represent the same real number iff $|M(n) - M'(m)| \leq 2^{-n} + 2^{-m}$ for all n, m.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	?	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	?
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

Markov's principle states: $\forall f : \mathbb{N} \to \mathbb{N}.\ \neg\neg(\exists n \in \mathbb{N}.\ f(n) = 0) \Rightarrow (\exists n \in \mathbb{N}.\ f(n) = 0)$.

"$\mathrm{Eff}(\mathrm{TM}) \models$ 6" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$ and given the promise that it is *not not* the case that f has a zero, determines a zero of f.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7	Countable choice holds.	✓	?	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7	Countable choice holds.	✓	✓ (always!)	?
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

Countable choice states: $\big(\forall x \in \mathbb{N}.\ \exists y \in A.\ \varphi(x,y)\big) \Rightarrow \big(\exists f : \mathbb{N} \to A.\ \forall x \in \mathbb{N}.\ \varphi(x, f(x))\big)$.

"Eff(TM) $\models$ 7" amounts to: There is a Turing machine which, given a Turing machine computing for every $x \in \mathbb{N}$ some $y \in A$ together with a witness of $\varphi(x,y)$, outputs a Turing machine computing a suitable choice function $\mathbb{N} \to A$.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
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2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7	Countable choice holds.	✓	✓ (always!)	✓ (always!)
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

Countable choice states: $\big(\forall x \in \mathbb{N}.\ \exists y \in A.\ \varphi(x,y)\big) \Rightarrow \big(\exists f : \mathbb{N} \to A.\ \forall x \in \mathbb{N}.\ \varphi(x, f(x))\big)$.

"$\mathrm{Eff}(\mathrm{TM}) \models$ 7" amounts to: There is a Turing machine which, given a Turing machine computing for every $x \in \mathbb{N}$ some $y \in A$ together with a witness of $\varphi(x,y)$, outputs a Turing machine computing a suitable choice function $\mathbb{N} \to A$.

The effective topos

	statement	in Set	in Eff(TM)	in Eff(ITTM)
1	Every number is prime or not prime.	✓ (trivially)	✓	✓
2	Beyond every number there is a prime.	✓	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6	Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7	Countable choice holds.	✓	✓ (always!)	✓ (always!)
8	There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	✗	?

The effective topos

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Curious size phenomena

$\mathrm{Eff}(\mathrm{ITTM}) \models$ "There exists an injection $\mathbb{R} \hookrightarrow \mathbb{N}$."

means:

There is an infinite time Turing machine which inputs the source of an infinite time Turing machine A representing a real number and outputs a natural number $n(A)$ such that $n(A) = n(B)$ if and only if A and B represent the same real.

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This statement is witnessed by following infinite time Turing machine:

> Read the source of an infinite time Turing machine A from the tape.
> Simulate all infinite time Turing machines in a dovetailing fashion.
> As soon as a machine is found which represents the same real as A,
> output the index of this machine and halt.

Wrapping up

- Effective toposes are a good vehicle for studying the nature of computation, in particular its higher-order aspects.
- Effective toposes build links between constructive mathematics and programming, check out Laura's talk for more.
- Toposes allow for curious dream axioms.
- Toposes also have a geometric flavor:
  points, open and closed subtoposes, continuous maps between toposes, cohomology, …

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You are welcome to explore the toposophic landscape :-)