FLABBY AND INJECTIVE OBJECTS IN TOPOSES

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Abstract. We introduce a general notion of *flabby objects* in elementary toposes and study their basic properties. In the special case of localic toposes, this notion reduces to the established notion of flabby sheaves, yielding a site-independent characterization of flabby sheaves. Continuing a line of research started by Roswitha Harting, we use flabby objects to show that an internal notion of injective objects coincides with the corresponding external notion, in stark contrast with the situation for projective objects. We show as an application that higher direct images can be understood as internal cohomology, and we study flabby objects in the effective topos.

As is nowadays well-established, every topos supports an *internal language* which can be used to reason about the objects and morphisms of the topos in a naive element-based language, allowing us to pretend that the objects are plain sets (or types) and that the morphisms are plain maps between those sets ([14, Chapter 6], [15, Section 1.3], [20, Chapter 14], [35, Chapter VI]). The internal language is sound with respect to intuitionistic reasoning, whereby every intuitionistic theorem holds in every topos.

The internal language of a sheaf topos enables *relativization by internalization*. For instance, by interpreting the proposition

> "in every short exact sequence of modules, if the two outer ones are finitely generated then so is the middle one"

of intuitionistic commutative algebra internally to the topos of sheaves over a space X, we obtain the geometric analogue

> "in every short exact sequence of sheaves of modules over X, if the two outer ones are of finite type then so is the middle one".

This way of deducing geometric theorems provides conceptual clarity, reduces technical overhead and justifies certain kinds of "fast and loose reasoning" typical of informal algebraic geometry. As soon as we go beyond the fragment of geometric sequents and consider more involved first-order or even higher-order statements, also significant improvements in proof length and proof complexity can be obtained. For instance, Grothendieck's generic freeness lemma admits a short and simple proof in this framework, while previously-published proofs proceed in a somewhat involved series of reduction steps and require a fair amount of prerequisites in commutative algebra [13, 11].

The practicality of this approach hinges on the extent to which the dictionary between internal and external notions has been worked out. For instance, the simple example displayed above hinges on the dictionary entry stating that a sheaf of modules is of finite type if and only if it looks like a finitely generated module from the internal point of view. The motivation for this note was to find internal characterizations of flabby sheaves and of higher direct images, and the resulting entries are laid out in Section 3 and in Section 4: A sheaf is flabby if and only if, from the internal point of view, it is a flabby set, a notion introduced in Section 2 below; and higher direct images look like sheaf cohomology from the internal point of view.

As a byproduct, we demonstrate how the notion of flabby sets is a useful organizing principle in the study of injective objects. We employ flabby sets to give a new proof of Roswitha Harting's results that injectivity of sheaves is a local notion [22] and that a sheaf is injective if and only if it is injective from the internal point of view [25], which she stated (in slightly different language) for sheaves of abelian groups. We use the opportunity to correct a small inaccuracy of hers, namely claiming that the analogous results for sheaves of modules would be false.

When employing the internal language of a topos, we are always referring to Mike Shulman's extension of the usual internal language, his *stack semantics* [46]. This extension allows to internalize unbounded quantification, which among other things is required to express the internal injectivity condition and the internal construction of sheaf cohomology via injective resolutions.

A further motivation for this note was our desire to seek a constructive account of sheaf cohomology. Sheaf cohomology is commonly defined using injective resolutions, which can fail to exist in the absence of the axiom of choice [9], but flabby resolutions can also be used in their stead, making them the obvious candidate for a constructively sensible replacement of the usual definition. However, we show in Section 5 and in Section 6 that flabby resolutions present their own challenges, and in summary we failed to reach this goal. The problem of giving a constructive account of sheaf cohomology is still open [49].

In view of almost 80 years of sheaf cohomology, this state of affairs is slightly embarrassing, challenging the call that "once [a] subject is better understood, we can hope to refine its definitions and proofs so as to avoid [the law of excluded middle]" [50, Section 3.4].

A constructive account of sheaf cohomology would be highly desirable, not only out of a philosophical desire to obtain a deeper understanding of the foundations of sheaf cohomology, but also to: use the tools of sheaf cohomology in the internal setting of toposes, thereby extending their applicability by relativization by internalization; and to carry out *integrated developments* of algorithms for computing sheaf cohomology, where we would extract algorithms together with termination and correctness proofs from a hypothetical constructive account.

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1. Flabby sheaves

A sheaf F on a topological space or a locale X is *flabby* (flasque) if and only if all restriction maps $F(X) \to F(U)$ are surjective. The following properties of flabby sheaves render them fundamental to the theory of sheaf cohomology:

(1) Let $(U_i)_i$ be an open covering of X. A sheaf F on X is flabby if and only if all of its restrictions $F|_{U_i}$ are flabby as sheaves on U_i.

(2) Let $f : X \to Y$ be a continuous map. If F is a flabby sheaf on X, then $f_*(F)$ is a flabby sheaf on Y.
(3) Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves of modules.
  (a) If F is flabby, then this sequence is also exact as a sequence of presheaves.
  (b) If F and H are flabby, then so is G.
  (c) If F and G are flabby, then so is H.
(4a) Every sheaf embeds canonically into a flabby sheaf.
(4b) Every sheaf of modules embeds canonically into a flabby sheaf of modules.

Since we want to develop an analogous theory for flabby objects in elementary toposes, it is worthwhile to analyze the logical and set-theoretic commitments which are required to establish these properties. The standard proofs of properties (1), (3a), (3b) and (3c) require Zorn's lemma to construct maximal extensions of given sections.[1] The standard proof of property (4b) requires the law of excluded middle, to ensure that the Godement construction actually yields a flabby sheaf. Properties (2) and (4a) can be verified purely intuitionistically.

There is an alternative definition of flabbiness, to be introduced below, which is equivalent to the usual one in presence of Zorn's lemma and which requires different commitments: For the alternative definition, properties (1), (3b), (4a) and (4b) can be verified purely intuitionistically. There is a substitute for property (3a) which can be verified purely intuitionistically.

Both definitions can be generalized to yield notions of flabby objects in elementary toposes; but for toposes which are not localic, the two resulting notions will differ, and only the one obtained from the alternative definition is stable under pullback and can be characterized in the internal language. We therefore adopt in this paper the alternative one as the official definition.

Definition 1.1. A sheaf F on a topological space (or locale) X is *flabby* if and only if for all opens U and all sections $s \in F(U)$, there is an open covering $X = \bigcup_i U_i$ such that, for all i, the section s can be extended to a section on $U \cup U_i$.[2]

If F is a flabby sheaf in the traditional sense, then F is obviously also flabby in the sense of Definition 1.1 – the singleton covering $X = X$ will do. Conversely, let F be a flabby sheaf in the sense of Definition 1.1. Let $s \in F(U)$ be a local section. Zorn's lemma implies that there is a maximal extension $s' \in F(U')$. By assumption, there is an open covering $X = \bigcup_i U_i$ such that, for all i, the section s' can be extended to $U' \cup U_i$. Since s' is maximal, $U' \cup U_i = U'$ for all i. Therefore $X = \bigcup_i U_i \subseteq U'$; hence s' is a global section, as desired.

We remark that unlike the traditional definition of flabbiness, Definition 1.1 exhibits flabbiness as a manifestly local notion.

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[1]We are careful to distinguish between the axiom of choice and Zorn's lemma. The former implies the latter, but the converse implication requires the law of excluded middle.

[2]Even in the absence of the axiom of choice it doesn't make a difference whether we are stipulating, as in the definition, for each index $i \in I$ the mere existence of an extension $t \in F(U \cup U_i)$, or whether we are stipulating the existence of a family $(t_i)_i$ of extensions $t_i \in F(U \cup U_i)$. Clearly, the existence of the family implies the existence of individual extensions. Conversely, if for each $i \in I$ there exists an extension, there is a tautologous family of extensions over the enlarged index set $I' := \{(i,t) \mid i \in I, t \in F(U \cup U_i), t|_U = s\}$, and we still have $X = \bigcup_{(i,t) \in I'} U_i$.

2. Flabby sets

We intend this section to be applied in the internal language of an elementary topos; we will speak about sets and maps between sets, but intend our arguments to be applied to objects and morphisms in toposes. We will therefore be careful to reason purely intuitionistically. We adopt the terminology of [34] regarding subterminals and subsingletons: A subset $K \subseteq X$ is *subterminal* if and only if every given elements are equal ($\forall x, y \in K.\ x = y$), and it is a *subsingleton* if and only if there is an element $x \in X$ such that $K \subseteq \{x\}$. Every subsingleton is trivially subterminal, but the converse might fail.

Definition 2.1. A set X is *flabby* if and only if every subterminal subset of X is a subsingleton, that is, if and only if for every subset $K \subseteq X$ such that $\forall x, y \in K.\ x = y$, there exists an element $x \in X$ such that $K \subseteq \{x\}$.

In the presence of the law of excluded middle, a set is flabby if and only if it is inhabited. This characterization is a constructive taboo:

Proposition 2.2. *If every inhabited set is flabby, then the law of excluded middle holds.*

Proof. Let φ be a truth value. The set $X := \{0\} \cup \{1 \,|\, \varphi\} \subseteq \{0, 1\}$ is inhabited by 0 and contains 1 if and only if φ holds. Let K be the subterminal $\{1 \,|\, \varphi\} \subseteq X$. Flabbiness of X implies that there exists an element $x \in X$ such that $K \subseteq \{x\}$. We have $x \neq 1$ or $x = 1$. The first case entails $\neg\varphi$. The second case entails $1 \in X$, so φ. □

Let $\mathcal{P}_{\leq 1}(X)$ be the set of subterminals of X.

Proposition 2.3. *A set X is flabby if and only if the canonical map $X \to \mathcal{P}_{\leq 1}(X)$ which sends an element x to the singleton set $\{x\}$ is final.*

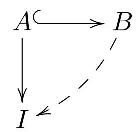
Proof. By definition. □

The set $\mathcal{P}_{\leq 1}(X)$ of subterminals of X can be interpreted as the set of *partially-defined elements* of X. In this view, the empty subset is the maximally undefined element and a singleton is a maximally defined element. A set is flabby if and only if every of its partially-defined elements can be refined to an honest element.

Remark 2.4. A set X is *$\neg\neg$-separated* if and only if $\neg\neg(x = y) \Rightarrow x = y$ for all elements $x, y \in X$. Although Section 2.1 presents some relation between flabby sets and $\neg\neg$-separated sets, neither notion encompasses the other. The set Ω is flabby, but might fail to be $\neg\neg$-separated; the set $\mathbb{Z}$ is $\neg\neg$-separated, even discrete, but might fail to be flabby. This can abstractly be seen by adapting the proof of Proposition 2.2. An explicit model in which $\mathbb{Z}$ is not flabby can be obtained by picking any topological space T such that $H^1(T, \underline{\mathbb{Z}}) \neq 0$, where $\underline{\mathbb{Z}}$ is the constant sheaf with stalks $\mathbb{Z}$. For instance, the space T could be connected (such that every global section is constant) while having an open which is the disjoint union of two connected components. Then the sheaf $\underline{\mathbb{Z}}$ is not flabby and hence, by Proposition 3.3 below, not a flabby set from the internal point of view of $\mathrm{Sh}(T)$.

Definition 2.5. (1) A set I is *injective* if and only if, for every injection $i : A \to B$, every map $f : A \to I$ can be extended to a map $B \to I$.

(2) An R-module I is *injective* if and only if, for every linear injection $i : A \to B$ between R-modules, every linear map $f : A \to I$ can be extended to a linear map $B \to I$, as in the diagram below.



In the presence of the law of excluded middle, a set is injective if and only if it is inhabited. In the presence of the axiom of choice, an abelian group is injective (as a $\mathbb{Z}$-module) if and only if it is divisible. Injective sets and modules have been intensively studied in the context of foundations before [9, 25, 33, 1]; the following properties are well-known:

Proposition 2.6. (1) *Every set embeds canonically (that is, in a uniform fashion) into an injective set.*
(2) *Every injective module is also injective as a set.*
(3) *Assuming the axiom of choice (so Zorn's lemma in combination with the law of excluded middle), every module embeds into an injective module.*

Proof. (1) One can check that, for instance, the full powerset $\mathcal{P}(X)$ and the set of subterminals $\mathcal{P}_{\leq 1}(X)$ are each injective.
(2) This statement follows from general abstract nonsense, since the forgetful functor from modules to sets possesses a monomorphism-preserving left adjoint. More explicitly, let I be an injective R-module, let $i : A \to B$ be an injective map between sets and let $f : A \to I$ be an arbitrary map. Then the induced map $R\langle A\rangle \to R\langle B\rangle$ between free modules is also injective, the given map f lifts to a linear map $R\langle A\rangle \to I$, and an R-linear extension $R\langle B\rangle \to I$ induces an extension $B \to I$ of f.
(3) One verifies that every abelian group embeds into a divisible abelian group. By Baer's criterion (which in this form requires the axiom of choice), divisible abelian groups are injective. The result for modules over arbitrary rings then follows purely formally, since the functor $A \mapsto \mathrm{Hom}(R, A)$ from abelian groups to R-modules has a left exact left adjoint with monic unit. □

We note in passing that the multi-step technique of the proof of Theorem 2.12 below can be used to verify Proposition 2.6(3) also in the absence of the law of excluded middle (but still requiring Zorn's lemma). Since this observation is of no further import for the purposes of this text, details are postponed to Section A.

Proposition 2.7. *Every injective set is flabby.*

Proof. Let I be an injective set. Let $K \subseteq I$ be a subterminal. The inclusion $f : K \to I$ extends along the injection $K \to 1 = \{\star\}$ to a map $1 \to I$. The unique image x of that map has the property that $K \subseteq \{x\}$. □

Corollary 2.8. *Every set canonically embeds into a flabby set.*

Proof. Immediate by Proposition 2.6(1) and Proposition 2.7. □

A further corollary of Proposition 2.7 is that the statement "every inhabited set is injective" is a constructive taboo: If every inhabited set is injective, then every inhabited set is flabby, thus the law of excluded middle follows by Proposition 2.2.

Proposition 2.9. *Every singleton set is flabby. The binary cartesian product of flabby sets is flabby.*

Proof. Immediate. □

Subsets of flabby sets are in general not flabby, as else every set would be flabby in view of Corollary 2.8.

2.1. **On the existence of enough flabby modules.** For the intended application to the theory of sheaf cohomology, the existence of *enough flabby sheaves of modules* is crucial: Any sheaf of modules embeds into a flabby sheaf of modules. In this section, we study the existence of enough flabby modules from the non-sheaf theoretic but constructive point of view.

We have already established that every set embeds into a flabby set. However, the supersets suggested by the proof of Corollary 2.8 do not carry a module structure even when the base set does. This deficit raises the following question: Given a module M, does there exist a linear injection $M \to M'$ into a set M' which simultaneously carries a module structure and is flabby?

An earlier version of this text formulated this question as an open problem. The positive solution presented below rests on the intermediate notion of *functionally flabby* sets and makes essential use of sheaves for modal operators. A short survey for the latter is contained in [13, Section 6]; other references include [20, Sections 14.4f.], [19] [53] and [18].[3]

Definition 2.10. A set X is *functionally flabby* if and only if there is a map $\varepsilon : \mathcal{P}_{\leq 1}(X) \to X$ such that for all $K \in \mathcal{P}_{\leq 1}(X)$, $K \subseteq \{\varepsilon(K)\}$.

Trivially, every functionally flabby set is flabby in the sense of Definition 2.1. A map $\varepsilon : \mathcal{P}_{\leq 1}(X) \to X$ satisfies the condition in Definition 2.10 if and only if it is a retraction of the canonical injection $X \to \mathcal{P}_{\leq 1}(X)$.

Proposition 2.11. *For a set X, the following conditions are equivalent:*

(1) *X is injective.*
(2) *X is functionally flabby.*

Proof. Let X be an injective set. Then the identity $X \to X$ can be extended along the canonical inclusion $X \to \mathcal{P}_{\leq 1}(X)$ to a map $\varepsilon : \mathcal{P}_{\leq 1}(X) \to X$. This map witnesses that X is functionally flabby.

Conversely, let X be a functionally flabby set with witness $\varepsilon : \mathcal{P}_{\leq 1}(X) \to X$. Let $i : A \to B$ be an injection and let $f : A \to X$ be a map. Then $x \mapsto \varepsilon(f[i^{-1}[\{x\}])$ is an extension of f along i. □

Theorem 2.12. *Every module embeds canonically into a functionally flabby module.*

Proof. We explain how to construct the required flabby envelopes in three steps. First, let M be a module which is a sheaf for some modal operator ∇ with the property that for any formula φ, $\nabla(\varphi \vee (\varphi \Rightarrow \nabla\bot))$. In this case, the module M is already functionally flabby: A witnessing choice function maps a subterminal $K \subseteq M$ to the unique element $x \in M$ such that $\nabla(x \in K')$, where $K' := K \cup \{0 \,|\, K \text{ inhabited} \Rightarrow \nabla\bot\}$. Since the set K' has, unlike K, the

[3] On page 5 of the preprint [53] there is a slight typing error: Fact 2.1(i) gives the characterization of j-closedness, not j-denseness. The correct characterization of j-denseness in that context is $\forall b \in B.\ j(b \in A)$.

property that $\nabla(K'$ is a singleton), such an element x exists and is unique by the sheaf condition.

Second, let M be a module which is separated for some modal operator ∇ as above. Then its sheafification is a functionally flabby module into which M embeds since the canonical map from M to its sheafification is injective by separatedness of M.

Finally, let M be an arbitrary module. For any element $x \in M$, let ∇_x be the modal operator with $\nabla_x \varphi :\equiv ((\varphi \Rightarrow x = 0) \Rightarrow x = 0)$. This is a modal operator of the kind as above. The module M might not be ∇_x-separated for any particular element x, but it is jointly so: For all elements $a, b \in M$, if $\nabla_x(a = b)$ for all $x \in M$, then $a = b$ (considering $x := a - b$). Hence the canonical map $M \to \prod_{x \in M} M^{+_x}$ into the product of the plus constructions with respect to all the modal operators ∇_x is injective, and the desired injection is the composition

$$M \longrightarrow \prod_{x \in M} M^{+_x} \longrightarrow \prod_{x \in M} M^{+_x +_x}$$

into the product of the sheafifications. □

Remark 2.13. For the purpose of verifying that any module embeds into a flabby module, it is essential that in the first step of the proof of Theorem 2.12 the module M is not only shown to be flabby, but even functionally flabby, and moreover in an explicit manner, explicitly presenting a witnessing map $\mathcal{P}_{\leq 1}(M) \to M$. This is because while an arbitrary product of flabby sets can fail to be flabby, an arbitrary product of functionally flabby sets with given witnesses is again so.

Given the somewhat nontrivial nature of the construction in the proof of Theorem 2.12, a natural question is whether simpler constructions exist as well. There are a number of simple constructions which come close to providing flabby envelopes for arbitrary modules, but all such constructions known to the author fail in some manner. For instance, given a module M, we could equip the set $T := \mathcal{P}_{\leq 1}(M)/\sim$, where $K \sim L$ if and only if $K = L$ or $K \cup L \subseteq \{0\}$, with a module structure given by $0 := [\{0\}]$, $[K] + [L] := [K + L]$ and $r[K] := rK$. The resulting module admits a linear injection from M, sending an element x to $[\{x\}]$. However, it fails to be flabby. Given a subterminal $E \subseteq \mathcal{P}_{\leq 1}(M)/\sim$, there is the well-defined element $v := [\{x \in M \mid x \in K \text{ for some } [K] \in E\}]$, but we cannot verify $E \subseteq \{v\}$.

That said, there is an alternative construction if sufficiently general *quotient inductive types*, as suggested by Altenkirch and Kaposi [2], are available. These generalize ordinary inductive W-types, which exist in any topos [42, 8, 7] and whose existence can indeed be verified in an intuitionistic set theory like IZF [17], by allowing to give constructors and state identifications at the same time. More specifically, given an R-module M, we can construct a flabby envelope T of M as the quotient inductive type generated by the following clauses, starting out as the construction of the free module over the underlying set of M: $0 \in T$ (where 0 is a formal symbol); if $t, s \in T$, then $t + s \in T$; if $t \in T$ and $r \in R$, then $rt \in T$; if $x \in M$, then $\underline{x} \in T$; if $K : I \to T$ is a family of elements of T indexed by a subterminal, then $\varepsilon_K \in T$; if $t, s, u \in T$ and $r, r' \in R$, then $t + (s + u) = (t + s) + u$, $t + s = s + t$, $t + 0 = t = t + 0$, $0t = 0$, $1t = t$, $r(t + u) = rt + ru$, $(r + r')t = rt + r't$; if $x, y \in M$ and $r \in R$, then $\underline{0} = 0$, $\underline{x + y} = \underline{x} + \underline{y}$, $\underline{rx} = r\underline{x}$; and if $K : I \to T$ is a family such that I is inhabited by some element i_0, then $\varepsilon_K = K(i_0)$.

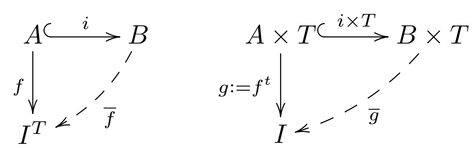
However, there are two issues with this approach. Firstly, it is an open question under which circumstances quotient inductive types can be shown to exist. Zermelo–Fraenkel with choice certainly suffices, while Zermelo–Fraenkel without choice does not [47, Section 9], hence IZF also does not.[4] The existence of quotient inductive types seems to be, as the existence of enough injective modules, *constructively neutral*. Secondly, by referencing arbitrary families of elements, the construction transcends the given type-theoretic or set-theoretic universe; the resulting object T is not manifestly small even if M is.

2.2. **Exactness properties.**

Proposition 2.14. (1) *Let I be an injective set. Let T be an arbitrary set. Then the set I^T of maps from T to I is injective.*

(2) *Let I be an injective R-module. Let T be an arbitrary R-module. If T is flat, then the module $\mathrm{Hom}_R(T, I)$ of linear maps from T to I is injective. In the general case, it is at least flabby.*

Proof. The first claim follows abstractly from the fact that the Hom functor $(\cdot)^T$ has a monomorphism-preserving left adjoint, namely the product functor $(\cdot) \times T$. Explicitly, an extension problem as in the left half of the diagram



can be transposed to the extension problem as in the right half. A solution $\overline{g}$ gives rise to the solution $\overline{f}$ of the original problem by the setting $\overline{f}(x) = (t \mapsto \overline{g}(x, t))$.

The injectivity part of the second claim follows entirely analogously, employing the tensor product instead of the cartesian product.

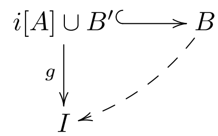
For the general statement, let K be a subterminal of $\mathrm{Hom}_R(T, I)$. Let T' be the submodule $\{s \in T \mid s = 0 \text{ or } K \text{ is inhabited}\} \subseteq T$ and let $f : T' \to I$ be the linear map defined as follows: Let $s \in T'$. If $s = 0$, then we set $f(s) = 0$; if K is inhabited, then we set $f(s) := g(s)$, where g is an arbitrary element of K. This association is well-defined. Since I is injective as a module, there is a linear extension $\overline{f} : T \to I$ of f along the inclusion $T' \subseteq T$. If K is inhabited, this extension is an element of K as required. □

Lemma 2.15. (1) *Let I be an injective set. Let $i : A \to B$ be an injection. Let $f : A \to I$ be an arbitrary map. Then the set of extensions of f to B is flabby.*

(2) *Let I be an injective R-module. Let $i : A \to B$ be a linear injection. Let $f : A \to I$ be an arbitrary linear map. Then the set of linear extensions of f to B is flabby.*

[4] With quotient inductive types, every infinitary algebraic theory admits free algebras. However, it is consistent with Zermelo–Fraenkel set theory that some such theories do not admit free algebras [10].

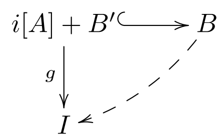
Proof. For the first claim, we set $X := \{\bar{f} \in I^B \mid \bar{f} \circ i = f\}$. Let $K \subseteq X$ be a subterminal. We consider the injectivity diagram



where B' is the set $\{s \in B \mid K \text{ is inhabited}\}$ and the solid vertical arrow g is defined in the following way: Let $s \in i[A] \cup B'$. If $s \in i[A]$, we set $g(s) := f(a)$, where $a \in A$ is an element such that $s = i(a)$. If $s \in B'$, we set $g(s) := \bar{f}(s)$, where $\bar{f}$ is any element of K. These prescriptions determine a well-defined map.

Since I is injective, there exists a dotted map rendering the diagram commutative. This map is an element of X. If K is inhabited, this map is an element of K.

The proof of the second claim is similar. We set $X := \{\bar{f} \in \mathrm{Hom}_R(B, I) \mid \bar{f} \circ i = f\}$. Let $K \subseteq X$ be a subterminal. We consider the injectivity diagram



where B' is the submodule $\{t \in B \mid t = 0 \text{ or } K \text{ is inhabited}\} \subseteq B$ and the solid vertical arrow g is defined in the following way: Let $s \in i[A] + B'$. Then $s = i(a) + t$ for an element $a \in A$ and an element $t \in B'$. Since $t \in B'$, $t = 0$ or K is inhabited. If $t = 0$, we set $g(s) := f(a)$. If K is inhabited, we set $g(s) := f(a) + \bar{f}(s)$, where $\bar{f}$ is any element of K. These prescriptions determine a well-defined map.

Since I is injective, there exists a dotted map rendering the diagram commutative. This map is an element of X. Furthermore, if K is inhabited, then this map is an element of K. □

Proposition 2.16. *Let* $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ *be a short exact sequence of modules. Let* $s \in M''$*. If* M' *is flabby, then the set of preimages of* s *under* p *is flabby.*

Proof. Let $X := \{u \in M \mid p(u) = s\}$. Let $K \subseteq X$ be a subterminal. Since p is surjective, there is an element $u_0 \in X$. The translated set $K - u_0 \subseteq M$ is still a subterminal, and its preimage under i is as well. Since M' is flabby, there is an element $v \in M'$ such that $i^{-1}[K - u_0] \subseteq \{v\}$. We verify that $K \subseteq \{u_0 + i(v)\}$:

Thus let $u \in K$ be given. Then $p(u - u_0) = 0$, so by exactness the set $i^{-1}[K - u_0]$ is inhabited. It therefore contains v. Thus $i(v) \in K - u_0$. Since $K = \{u\}$, it follows that $i(v) = u - u_0$, so $u \in \{u_0 + i(v)\}$ as claimed. □

Toby Kenney stressed that the notion of an injective set should be regarded as an interesting strengthening of the constructively rather ill-behaved notion of a nonempty set [33]. For instance, while the statements "there is a choice function for every set of nonempty sets" and even "there is a choice function for every set of inhabited sets" are constructive taboos, the statement "there is a choice function for every set of injective sets" is constructively neutral. Proposition 2.16 demonstrates that the notion of a flabby set can be regarded as an interesting intermediate notion: In the situation of Proposition 2.16, the set of preimages is not only not empty or inhabited, but even flabby.

Proposition 2.17. *Let* $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ *be a short exact sequence of modules. If* M' *and* M'' *are flabby, so is* M*.*

Proof. Let $K \subseteq M$ be a subterminal. Then its image $p[K] \subseteq M''$ is a subterminal as well. Since M'' is flabby, there is an element $s \in M''$ such that $p[K] \subseteq \{s\}$.

Since p is surjective, there is an element $u_0 \in M$ such that $p(u_0) = s$.

The preimage $i^{-1}[K - u_0] \subseteq M'$ is a subterminal. Since M' is flabby, there exists an element $v \in M'$ such that $i^{-1}[K - u_0] \subseteq \{v\}$.

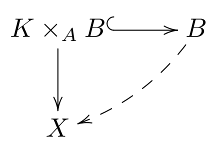
Thus $K \subseteq \{u_0 + i(v)\}$. □

Noticeably missing here is a statement as follows: "Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M are flabby, so is M''." Assuming Zorn's lemma in the metatheory, this statement is true in every topos of sheaves over a locale, but we do not know whether it has an intuitionistic proof and in fact we surmise that it has not.

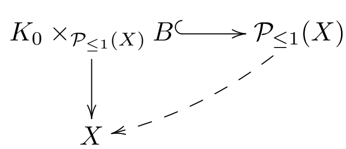
3. Flabby objects

Definition 3.1. An object X of an elementary topos $\mathcal{E}$ is *flabby* if and only if the statement "X is a flabby set" holds in the stack semantics of $\mathcal{E}$.

This definition amounts to the following: An object X of an elementary topos $\mathcal{E}$ is flabby if and only if, for every monomorphism $K \to A$ and every morphism $K \to X$, there exists an epimorphism $B \to A$ and a morphism $B \to X$ such that the following diagram commutes.



Instead of referring to arbitrary stages $A \in \mathcal{E}$, one can also just refer to the generic stage: Let $\mathcal{P}_{\leq 1}(X)$ denote the *object of subterminals* of X; this object is a certain subobject of the powerobject $\mathcal{P}(X) = [X, \Omega_\mathcal{E}]_\mathcal{E}$. The subobject K_0 of $X \times \mathcal{P}_{\leq 1}(X)$ classified by the evaluation morphism $X \times \mathcal{P}_{\leq 1}(X) \to X \times \mathcal{P}(X) \to \Omega_\mathcal{E}$ is the *generic subterminal* of X. The object X is flabby if and only if there exists an epimorphism $B \to \mathcal{P}_{\leq 1}(X)$ and a morphism $B \to X$ such that the following diagram commutes.



Proposition 3.2. *Let* X *and* T *be objects of an elementary topos* $\mathcal{E}$*.*

(1) *If* X *is flabby, so is* $X \times T$ *as an object of* $\mathcal{E}/T$*.*
(2) *The converse holds if the unique morphism* $T \to 1$ *is an epimorphism.*

Proof. This holds for every property which can be defined in the stack semantics [46, Lemma 7.3]. □

Proposition 3.3. *Let* F *be a sheaf on a topological space* X *(or a locale). Then* F *is flabby as a sheaf if and only if* F *is flabby as an object of the sheaf topos* $\mathrm{Sh}(X)$*.*

Proof. The proof is routine; we only verify the "only if" direction. Let F be flabby as a sheaf. It suffices to verify the defining condition for stages of the form $A = \mathrm{Hom}(\cdot, U)$, where U is an open of X. A monomorphism $K \to A$ then amounts to an open $V \subseteq U$ (the union of all opens on which K is inhabited). A morphism $K \to F$ amounts to a section $s \in F(V)$. Since F is flabby as a sheaf, there is an open covering $X = \bigcup_{i \in I} V_i$ such that, for all i, the section s can be extended to a section s_i of $V \cup V_i$. The desired epimorphism is $B := \coprod_i \mathrm{Hom}(\cdot, (V \cup V_i) \cap U) \to A$, and the desired morphism $B \to X$ is given by the sections $s_i|_{(V \cup V_i) \cap U}$. **XXX: choicefree**

As stated, the argument in the previous paragraph requires the axiom of choice to pick the extensions s_i; this can be avoided by a standard trick of expanding the index set of the coproduct to include the choices: We redefine $B := \coprod_{(i,t) \in I'} \mathrm{Hom}(\cdot, (V \cup V_i) \cap U)$, where $I' = \{(i \in I, t \in F(V \cup V_i)) \mid t|_V = s\}$ and define the morphism $B \to X$ on the (i,t)-summand by $t|_{(V \cup V_i) \cap U}$. ☐

Example 3.4. The object of Dedekind reals of a topos is in general not flabby. For instance, in the case of the topos of sheaves over the real line $\mathbb{R}^1$, the object of Dedekind reals is the sheaf $\mathcal{C}$ of continuous (Dedekind-)real valued functions **XXX: cite**. The section $1/x$ cannot be extended to opens containing the origin.

Proposition 3.5. *Let X be a flabby object of a localic topos $\mathcal{E}$. If Zorn's lemma is available in the metatheory, then X possesses a global element (a morphism $1 \to X$).*

Proof. This is a restatement of the discussion following Definition 1.1. ☐

Remark 3.6. Some condition on the topos is necessary for flabby objects to possess global elements. An example is given by the G-set G (with the translation action), considered as an object of the topos BG of G-sets, where G is a nontrivial group. This object is flabby (because it is inhabited and BG is a Boolean topos, assuming the law of excluded middle in the metatheory), but it does not have any global elements.

Proposition 3.7. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. If f_* preserves epimorphisms, then f_* preserves flabby objects.*

Proof. Let $X \in \mathcal{F}$ be a flabby object. Let $k : K \to A$ be a monomorphism in $\mathcal{E}$ and let $x : K \to f_*(X)$ be an arbitrary morphism. Without loss of generality, we may assume that A is the terminal object 1 of $\mathcal{E}$. Then $f^*(k) : f^*(K) \to 1$ is a monomorphism in $\mathcal{F}$ and the adjoint transpose $x^t : f^*(K) \to X$ is a morphism in $\mathcal{F}$. Since X is flabby, there is an epimorphism $B \to 1$ in $\mathcal{F}$ and a morphism $y : B \to X$ such that the morphism $f^*(K) \times B \to X$ factors over y. Hence x factors over $f_*(y) : f_*(B) \to f_*(X)$. We conclude because the morphism $f_*(B) \to f_*(1)$ is an epimorphism by assumption. ☐

The assumption on f_* of Proposition 3.7 is for instance satisfied if f is a local geometric morphism.

Remark 3.8. Pullbacks of flabby objects along geometric morphisms are usually not flabby. For instance, constant sheaves can fail to be flabby (Remark 2.4) but arise as pullbacks along geometric morphisms to the topos Set, in which most objects are flabby (assuming the law of excluded middle).

Definition 3.9. An object I of an elementary topos $\mathcal{E}$ is *externally injective* if and only if for every monomorphism $A \to B$ in $\mathcal{E}$, the canonical map $\mathrm{Hom}_{\mathcal{E}}(B, I) \to \mathrm{Hom}_{\mathcal{E}}(A, I)$ is surjective. It is *internally injective* if and only if for every monomorphism $A \to B$ in $\mathcal{E}$, the canonical morphism $[B, I] \to [A, I]$ between Hom objects is an epimorphism in $\mathcal{E}$.

If R is a ring in an elementary topos $\mathcal{E}$, a similar definition can be given for R-modules in $\mathcal{E}$, referring to the set respectively the object of linear maps. The condition for an object to be internally injective can be rephrased in various ways. The following proposition lists five of these conditions. The equivalence of the first four is due to Roswitha Harting [25].

Proposition 3.10. *Let $\mathcal{E}$ be an elementary topos. Then the following statements about an object $I \in \mathcal{E}$ are equivalent.*

(1) *I is internally injective.*
(1') *For every morphism $p : A \to 1$ in $\mathcal{E}$, the object $p^*(I)$ has property (1) as an object of $\mathcal{E}/A$.*
(2) *The functor $[\cdot, I] : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ maps monomorphisms in $\mathcal{E}$ to morphisms for which every global element of the target locally (after change of base along an epimorphism) possesses a preimage.*
(2') *For every morphism $p : A \to 1$ in $\mathcal{E}$, the object $p^*(I)$ has property (2) as an object of $\mathcal{E}/A$.*
(3) *The statement "I is an injective set" holds in the stack semantics of $\mathcal{E}$.*

Proof. The implications (1) $\Rightarrow$ (2), (1') $\Rightarrow$ (2'), (1') $\Rightarrow$ (1) and (2') $\Rightarrow$ (2) are trivial.

The equivalence (1') $\Leftrightarrow$ (3) follows directly from the interpretation rules of the stack semantics.

The implication (2) $\Rightarrow$ (2') employs the extra left adjoint $p_! : \mathcal{E}/A \to \mathcal{E}$ of $p^* : \mathcal{E} \to \mathcal{E}/A$ (which maps an object $(X \to A)$ to X), as in the usual proof that injective sheaves remain injective when restricted to smaller open subsets: We have that $p_* \circ [\cdot, p^*(I)]_{\mathcal{E}/A} \cong [\cdot, I]_{\mathcal{E}} \circ p_!$, the functor $p_!$ preserves monomorphisms, and one can check that p_* reflects the property that global elements locally possess preimages. Details are in [25, Thm. 1.1].[5]

The implication (2') $\Rightarrow$ (1') follows by performing an extra change of base, exploiting that any non-global element becomes a global element after a suitable change of base. □

Let R be a ring in $\mathcal{E}$. Then the analogue of Proposition 3.10 holds for R-modules in $\mathcal{E}$, if $\mathcal{E}$ is assumed to have a natural numbers object. The extra assumption is needed in order to construct the left adjoint $p_! : \mathrm{Mod}_{\mathcal{E}/A}(R \times A) \to \mathrm{Mod}_{\mathcal{E}}(R)$. Phrased in the internal language, this adjoint maps a family $(M_a)_{a \in A}$ of R-modules to the direct sum $\bigoplus_{a \in A} M_a$. Details on this construction, phrased in the language of sets but interpretable in the internal language, can for instance be found in [40, page 54].

Somewhat surprisingly, and in stark contrast with the situation for internally projective objects (which are defined dually), internal injectivity coincides with external injectivity for localic toposes. In the special case of sheaves of abelian groups, this result is due to Roswitha Harting [25, Proposition 2.1].

[5]Harting formulates her theorem for abelian group objects, and has to assume that $\mathcal{E}$ contains a natural numbers object to ensure the existence of an abelian version of $p_!$.

Theorem 3.11. *Let I be an object of an elementary topos $\mathcal{E}$. If I is externally injective, then I is also internally injective. The converse holds if $\mathcal{E}$ is localic and Zorn's lemma is available in the metatheory.*

Proof. Let I be an object which is externally injective. Then I satisfies Condition (2) in Proposition 3.10, even without having to pass to covers.

For the converse direction, let I be an internally injective object. Let $i : A \to B$ be a monomorphism in $\mathcal{E}$ and let $f : A \to I$ be an arbitrary morphism. We want to show that there exists an extension $B \to I$ of f along i. To this end, we consider the object of such extensions, defined by the internal expression

$$F := \{\bar{f} \in [B, I] \mid \bar{f} \circ i = f\}.$$

Global elements of F are extensions of the kind we are looking for. By Lemma 2.15(1), interpreted in $\mathcal{E}$, this object is flabby. By Proposition 3.5, it has a global element. □

The analogue of Theorem 3.11 for modules holds as well, if $\mathcal{E}$ is assumed to have a natural numbers object. The proof carries over word for word, only referencing Lemma 2.15(2) instead of Lemma 2.15(1). It seems that Roswitha Harting was not aware of this generalization, even though she did show that injectivity of sheaves of modules over topological spaces is a local notion [22, Remark 5], as she (mistakenly) states in [22, page 233] that "the notions of injectivity and internal injectivity do not coincide" for modules.

It is worth noting that, because the internal language machinery was at that point not as well-developed as it is today, Harting had to go to considerable length to construct internal direct sums of abelian group objects [24], and in order to verify that taking internal direct sums is faithful she appealed to Barr's metatheorem [23, Theorem 1.7]. Nowadays we can verify both statements by simply carrying out an intuitionistic proof in the case of the topos of sets and then trusting the internal language to obtain the generalization to arbitrary elementary toposes with a natural numbers object.

Since we were careful in Section 2 to use the law of excluded middle and the axiom of choice only where needed, most results of that section carry over to flabby and internally injective objects. Specifically, we have:

Scholium 3.12. *For every elementary topos $\mathcal{E}$:*

1. *Every object embeds canonically into an internally injective object.*
2. *(If $\mathcal{E}$ has a natural numbers object.) The underlying unstructured object of an internally injective module is internally injective.*
3. *Every internally injective object is flabby.*
4. *Every object embeds canonically into a flabby object.*
5. *Every internal module embeds canonically into an internal module which is flabby.*
6. *The terminal object is flabby. The binary product of flabby objects is flabby.*
7. *Let I be an internally injective object. Let T be an arbitrary object. Then $[T, I]$ is a flabby object.*
8. *(If $\mathcal{E}$ has a natural numbers object.) Let I be an internally injective R-module. Let T be an arbitrary R-module. Then $[T, I]_R$, the subobject of the internal Hom consisting only of the linear maps, is a flabby object.*
9. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules in $\mathcal{E}$. If M' and M'' are flabby objects, so is M.*

Proof. We established the analogous statements for sets and modules purely intuitionistically in Section 2, and the stack semantics is sound with respect to intuitionistic logic. □

Scholium 3.13. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules in a localic topos $\mathcal{E}$. Let M' be a flabby object. Assuming Zorn's lemma in the metatheory, the induced sequence $0 \to \Gamma(M') \to \Gamma(M) \to \Gamma(M'') \to 0$ of $\Gamma(R)$-modules is exact, where $\Gamma(X) = \mathrm{Hom}_{\mathcal{E}}(1, X)$.*

Proof. We only have to verify exactness at $\Gamma(M'')$, so let $s \in \Gamma(M'')$. Interpreting Proposition 2.16 in $\mathcal{E}$, we see that the object of preimages of s is flabby. Since $\mathcal{E}$ is localic, this object is a flabby sheaf; since Zorn's lemma is available, it possesses a global element. Such an element is the desired preimage of s in $\Gamma(M)$. □

If $\mathcal{E}$ is not necessarily localic or Zorn's lemma is not available, only a weaker substitute for Scholium 3.13 is available: Given $s \in \Gamma(M'')$, the object of preimages of s is flabby. In particular, given any point of $\mathcal{E}$, we can extend any local preimage of s to a preimage which is defined on an open neighborhood of that point. We believe that there are situations in which this weaker substitute is good enough, similar to how in constructive algebra often the existence of a sufficiently large field extension is good enough where one would classically blithely pass to an algebraic closure.

Remark 3.14. We can dispose the reliance on Zorn's lemma in Scholium 3.13 if instead $\mathcal{E}$ is compact. This is because XXX ... finitely many local preimages ... patch together ... by the following internal observation ...

Remark 3.15. A direct generalization of the traditional notion of a flabby sheaf, as opposed to our reimagining in Definition 1.1, to elementary toposes is the following. An object X of an elementary topos $\mathcal{E}$ is *strongly flabby* if and only if, for every monomorphism $K \to 1$ in $\mathcal{E}$, every morphism $K \to X$ lifts to a morphism $1 \to X$.

One can verify, purely intuitionistically, that a sheaf F on a space T is flabby in the traditional sense if and only if F is a strongly flabby object in $\mathrm{Sh}(T)$, if and only if F is a flabby object in the presheaf topos $\mathrm{PSh}(T)$.

The notion of strongly flabby objects is, however, not local (in the same sense that the notion of flabby objects is, as stated in Proposition 3.2) and therefore cannot be characterized in the internal language. A specific example is the G-set G (with the translation action) as in Remark 3.6. This object is not strongly flabby, since the morphism $\emptyset \to 1$ does not lift, but its pullback to the slice $BG/G \simeq \mathrm{Set}$ is (assuming the law of excluded middle in the metatheory), and the unique morphism $G \to 1$ is indeed an epimorphism.

4. Higher direct images as internal sheaf cohomology

Let X be a locale and let Y be a locale over X, that is, a morphism $f : Y \to X$ of locales. By the fundamental relation between locales and topological spaces, this situation arises for instance, when given a sober topological space and a topological space over it, as is often the case in algebraic topology or algebraic geometry. Let a sheaf $\mathcal{O}_Y$ of rings on Y be given. Then the traditional way to define the *higher direct images* of a sheaf E of $\mathcal{O}_Y$-modules is to pick an injective resolution $0 \to E \to I^\bullet$ and set $R^n f_*(E) := H^n(f_*(I^\bullet))$ [48, Tag 01DZ].

Assuming Zorn's lemma, there are enough injective sheaves of modules so that this recipe can be carried out. The resulting sheaf of $f_*\mathcal{O}_Y$-modules is well-defined in the following sense: Given a further injective resolution $0 \to E \to J^\bullet$, there is up to homotopy precisely one morphism $I^\bullet \to J^\bullet$ compatible with the identity on E, and this morphism induces an isomorphism on cohomology.

Higher direct images are pictured as a "relative" version of sheaf cohomology. Due to the result that injectivity of sheaves of modules can be characterized in the internal language, we can give a precise rendering of this slogan: *Higher direct images are internal sheaf cohomology.*

The details are as follows. The over-locale Y corresponds to a locale $I(Y)$ internal to $\mathrm{Sh}(X)$, in such a way that the category of internal sheaves over this internal locale coincides with $\mathrm{Sh}(Y)$ [31, Scholium C1.6.4]; in particular, a given sheaf E of $\mathcal{O}_Y$-modules can be regarded as a sheaf over $I(Y)$. Under this equivalence, the morphism $f : Y \to X$ corresponds to the unique morphism $I(Y) \to \mathrm{pt}$ to the internal one-point locale. Hence it makes sense to construct, from the internal point of view of $\mathrm{Sh}(X)$, the sheaf cohomology $H^n(I(Y), E)$ of E.

Usually one would not expect an internal construction which depends on arbitrary choices to yield a globally-defined sheaf over X – following the definition of the stack semantics we only obtain a family of sheaves defined on members of some open covering of X; but we verify in Theorem 4.2 below that in our case, it does, and that the resulting sheaf coincides with $R^n f_*(E)$.

Lemma 4.1. *Let Y be a ringed locale over a locale X. Let J be a sheaf of modules over Y. Assuming Zorn's lemma in the metatheory, the following statements are equivalent:*

1. *J is an injective sheaf of modules.*
2. *From the point of view of $\mathrm{Sh}(Y)$, J is an injective module.*
3. *From the point of view of $\mathrm{Sh}(X)$, J is an injective module from the point of view of $\mathrm{Sh}(I(Y))$.*
4. *From the point of view of $\mathrm{Sh}(X)$, J is an injective sheaf of modules on $I(Y)$.*

Proof. The equivalence of the first two statements is by Theorem 3.11. The equivalence (2) $\Leftrightarrow$ (3) is by the idempotency of the stack semantics: $\mathrm{Sh}(Y) \models \varphi$ if and only if $\mathrm{Sh}(X) \models (\mathrm{Sh}(I(Y)) \models \varphi)$. (Shulman stated and proved a restricted version of this idempotency property in his original paper on the stack semantics [46, Lemma 7.20]. A proof of the general case is slightly less accessible [12, Lemma 1.20].) The equivalence (3) $\Leftrightarrow$ (4) is by interpreting Theorem 3.11 internally to $\mathrm{Sh}(X)$. This requires Zorn's lemma to hold internally to $\mathrm{Sh}(X)$; this is indeed the case since we assume Zorn's lemma in the metatheory and the validity of Zorn's lemma passes from the metatheory to localic toposes [31, Proposition D4.5.14]. □

Theorem 4.2. *Let $f : Y \to X$ be a ringed locale over a locale X. Let E be a sheaf of modules over Y. Assuming Zorn's lemma in the metatheory, the expression "$H^n(I(Y), E)$" of the internal language of $\mathrm{Sh}(X)$ denotes a globally-defined sheaf on X, and this sheaf coincides with $R^n f_*(E)$.*

Proof. By Lemma 4.1 and by the fact that every sheaf of modules over Y admits an injective resolution, every sheaf of modules over $I(Y)$ admits an injective resolution from the point of view of $\mathrm{Sh}(X)$. Hence we can, internally to $\mathrm{Sh}(X)$, carry out the construction of $H^n(I(Y), E)$. Externally, this construction yields an open covering of X such that we have, for each member U of that covering

- a sheaf M over U,
- a module structure on M,
- a resolution $0 \to E|_{f^{-1}U} \to I^\bullet$ by sheaves of modules which are internally and hence externally injective and
- data exhibiting M as the n-th cohomology of $(f|_{f^{-1}U})_*(I^\bullet)$.

On intersections of such opens U and U', there is exactly one isomorphism $M|_{U\cap U'} \to M'|_{U\cap U'}$ of sheaves of modules induced by a morphism of resolutions which is compatible with the identity on E. Hence the cocycle condition for these isomorphisms is satisfied, ensuring that the individual sheaves M glue to a globally-defined sheaf of modules on X. (The individual injective resolutions need not glue to a global injective resolution.)

The claim that this sheaf coincides with $R^n f_*(E)$ follows from the fact that we can pick as internal resolution of E (considered as a sheaf over $I(Y)$) the particular injective resolution of E (considered as a sheaf over Y) used to define $R^n f_*(E)$. □

The internal characterization provided by Theorem 4.2 gives, as a simple application, a logical explanation of the basic fact that higher direct images along the identity $\mathrm{id} : X \to X$ vanish: From the internal point of view of $\mathrm{Sh}(X)$, the over-locale X corresponds to the one-point locale, and the higher cohomology of the one-point locale vanishes.

In algebraic geometry, the internal characterization can be used to immediately deduce the explicit description of the higher direct images of Serre's twisting sheaves along the projection $\mathbb{P}^n_S \to S$, where S is an arbitrary base scheme (or even base locally ringed locale), from a computation of the cohomology of projective n-space. Background on carrying out scheme theory internally to a topos is given in [13, Section 12].

5. Flabby objects in the effective topos

The notion of flabby objects originates from the notion of flabby sheaves and is therefore closely connected to Grothendieck toposes. Hence it is instructive to study flabby objects in elementary toposes which are not Grothendieck toposes, away from their original conceptual home. We begin this study with establishing the following observation on flabby objects in the effective topos. We follow the terminology of Martin Hyland's survey on the effective topos [29] and refer to [43, 41, 44, 6] for more background.

Proposition 5.1. *The only object in the effective topos which is both flabby and effective is the singleton object.*

The intuitive reason for why Proposition 5.1 holds is the following. Let X be a flabby object in the effective topos. Then there is a procedure which computes for any (realizer of a) subterminal $K \subseteq X$ a (realizer of an) element x_K such that $K \subseteq \{x_K\}$. However, realizers for subsets are not very informative; the procedure cannot ask questions such as "is a given element of X contained in K?" nor query its input in any way. Hence x_K will actually be the same element for any subterminal K. Metaphorically speaking, a procedure witnessing flabbiness has to conjure elements out of thin air.

This issue does not manifest with objects X which are not effective objects such as double-negation sheaves. Realizers for elements of such objects are themselves not very informative; for those, a procedure witnessing flabbiness only has to turn one kind of non-informative realizers into another kind. This is why, in line with Theorem 2.12, it is still true in the effective topos that any module of the effective topos embeds into a flabby module.

Proof of Proposition 5.1. Let X be an object of the effective topos. Then the object $P_{\leq 1}(X)$ of subterminals of X is a uniform object in the sense of [41, Section 3.4], being a retract of the uniform object $\mathcal{P}(X)$ by the surjection

$$\mathcal{P}(X) \longrightarrow \mathcal{P}_{\leq 1}(X),\ M \longmapsto \{x \in X \mid M = \{x\}\}.$$

Hence if X is effective, the uniformity principle [29, Proposition 15.1]

$$\forall K \in \mathcal{P}_{\leq 1}(X).\ \exists x \in X.\ K \subseteq \{x\} \quad \Longrightarrow \quad \exists x_0 \in X.\ \forall K \in \mathcal{P}_{\leq 1}(X).\ K \subseteq \{x_0\}$$

applies. Thus, if X is effective and flabby, there is an element $x_0 \in X$ such that $K \subseteq \{x_0\}$ for any subterminal K of X. The conclusion follows by considering, for any elements $a, b \in X$, the subterminals $\{a\}$ and $\{b\}$. □

Remark 5.2. The analogue of Proposition 5.1 is true as well for the realizability topos constructed using infinite time Turing machines [5, 21] and indeed for any realizability topos, with the same proof, as power objects are always uniform [30].

6. Conclusion

We originally set out to develop an intuitionistic account of Grothendieck's sheaf cohomology. Čech methods can be carried out constructively, and there are constructive accounts of special cases, resulting even in efficient-in-practice algorithms [3, 4], but there is no established general framework for sheaf cohomology which would work in an intuitionistic metatheory.

The main obstacle preventing Grothendieck's theory of derived functors to be interpreted constructively is its reliance on injective resolutions. It is known that in the absence of Zorn's lemma, much less in a purely intuitionistic context, there might not be any nontrivial injective abelian group [9].

Can this issue be remedied by employing flabby resolutions instead of injective ones? Classically, it is known that they can be used in their stead, and moreover George Kempf developed the foundations of the cohomology of quasicoherent sheaves on this premise [32]. There are, however, three issues with this suggestion.

Firstly, all proofs known to us that flabby sheaves are acyclic for the global sections functor require Zorn's lemma. This problem might be mitigated by relying on the substitute property discussed following Scholium 3.13, but still acyclitity in the usual sense will most likely not be attainable.

Secondly, while we have shown in Theorem 2.12 that enough flabby envelopes exist constructively, the analysis in Section 5 demonstrates that the presented envelopes are in a sense deeply uncomputable. Intuitionistic Zermelo–Fraenkel set theory [17] and the type theory of toposes [37] do verify that they exist, but only by virtue of excessive reliance on powersets, precluding concrete computations; the construction is highly *impredicative*, not meeting the bar of *predicative mathematics* [16].

As a result, thirdly, flabby resolutions do not generalize from toposes to the setting of *arithmetic universes*, the predicative cousins of toposes introduced by André Joyal which have recently been an important object of consideration by Milly Maietti and Steve Vickers [36, 38, 52]. These are not only interesting on their own and as a convenient foundation for ensuring predicativity (and hence computability in a strict sense), but also because they can be used to obtain base-independent proofs for the topos case [51, 26]: *Computing cohomology of concrete spaces should not require fixing a universe of sets first.*

We currently believe that it is not possible to give a constructive account of a global cohomology functor which would associate to any sheaf of modules its cohomology. However, it should be possible to do so for a restricted class of sheaves, while still preserving the good formal properties expected from derived functors. To this end, a mix of the approaches using pointwise Kan extensions, as cogently argued for by Emily Riehl [45, Chapter 2] (see also [27, 39]), and ind-objects as presented in the Stacks Project [48, Tag 05S7] seems promising.

This framework is sufficiently flexible to not demand derived functors to be defined everywhere. Rather, they will be defined just for those objects for which we happen to have a suitable resolution. Only classically, by using injective or flabby sheaves, can we pretend that every object has such a resolution. We hope to report on details in future work. Can perhaps even the rare cases where injective resolutions exceptionally do have some link to computations, as on projective spaces [28], be given a proper constructive and predicative home?

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Appendix A. On the existence of enough injective modules

Definition A.1. (1) An abelian group I is *divisible* if and only if for every element $x \in I$ and every natural number $n \geq 1$, there is an element $y \in I$ such that $ny = x$.
(2) An abelian group I *satisfies the Baer condition* if and only if, for every ideal $\mathfrak{a} \subseteq \mathbb{Z}$, every ($\mathbb{Z}$-)linear map $\mathfrak{a} \to I$ admits an extension along the inclusion to a linear map $\mathbb{Z} \to I$.

Every group satisfying the Baer condition is divisible, by specializing $\mathfrak{a}$ to the ideal (n) and using multiplication by x as the map $\mathfrak{a} \to I$.

Proposition A.2. (1) *Assuming the law of excluded middle, all divisible abelian groups satisfy the Baer condition.*
(2) *Assuming Zorn's lemma, all abelian groups which satisfy the Baer condition are injective.*

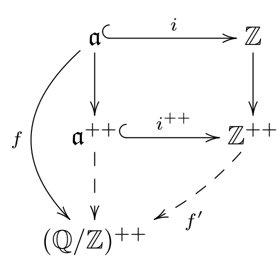
Proof. Assuming the law of excluded middle, the only ideals of $\mathbb{Z}$ are the principal ideals (n) where $n \geq 0$. For $n = 0$ existence of extensions is trivial and for $n > 0$ existence of extensions follow from divisibility.

Assuming Zorn's lemma, let I be an abelian group satisfying the Baer condition, let $i : A \to B$ be a linear injection and let $f : A \to I$ be a linear map. The poset of partial extensions of f contains suprema of chains and hence a maximal partial extension $f_0 : A_0 \to I$ with $A \subseteq A_0 \subseteq B$.

To verify that $A_0 = B$, let an element $x \in B$ be given. By the Baer condition, the linear map $g : (A_0 : x) \to I$ defined on the ideal $(A_0 : x) = \{n \in \mathbb{Z} \mid nx \in A_0\} \subseteq \mathbb{Z}$ given by $g(n) = f_0(nx)$ can be extended to a linear map $\overline{g} : \mathbb{Z} \to I$. Hence f_0 can be extended to the map $A_0 + (x) \to I$ given by $u + nx \mapsto f_0(u) + \overline{g}(n)$. By maximality $A_0 + (x) = A_0$, hence $x \in A_0$. □

Lemma A.3. *Let ∇ be a modal operator such that $\nabla(\varphi \vee (\varphi \Rightarrow \nabla\bot))$ and let $(\cdot)^+$ denote the plus construction with respect to ∇. Then:*
(1) *The sheafification $(\mathbb{Q}/\mathbb{Z})^{++}$ satisfies the Baer condition.*
(2) *Assuming Zorn's lemma, given an element x of an abelian group A, there is a linear map $g : A \to (\mathbb{Q}/\mathbb{Z})^{++}$ such that $g(x) = 0 \Rightarrow \nabla(x = 0)$.*

Proof. Let $i : \mathfrak{a} \hookrightarrow \mathbb{Z}$ be the inclusion of an ideal and let $f : \mathfrak{a} \to (\mathbb{Q}/\mathbb{Z})^{++}$ be a linear map as in the solid part of the following commutative diagram:



The given map f factors (uniquely) over the map $\mathfrak{a} \to \mathfrak{a}^{++}$ since $(\mathbb{Q}/\mathbb{Z})^{++}$ is a sheaf. The map i^{++} is still injective as sheafification is exact. (The two top vertical maps need not be injective, as $\mathbb{Z}$ need not be separated for ∇.) Hence we are reduced to an extension problem in the subtopos of ∇-sheaves, which is solvable because this subtopos is Boolean (Proposition A.2(1)). More explicitly, the extension f' maps $1 \in \mathbb{Z}$ to the gluing of

$$\{0 \,|\, \forall x \in \mathfrak{a}.\nabla(x = 0)\} \cup \\ \{\tfrac{1}{d}v \,|\, d \geq 1, v \in [0,1), f(d) = \underline{v}, \forall x \in \mathfrak{a}.\nabla(x \in \mathfrak{a}) \Leftrightarrow \nabla(\exists n \in \mathbb{Z}.x = nd)\}.$$

For the second part, let A be an abelian group and let $x \in A$ be an element. Let $i : \mathbb{Z}/(0 : x) \to A$ be the linear map with $[1] \mapsto x$, where $(0 : x)$ is the ideal $\{n \in \mathbb{Z} \,|\, nx = 0\}$. In the Boolean subtopos of ∇-sheaves, this ideal is either the zero ideal or the principal ideal (d) of some positive generator. Let $h : \mathbb{Z}/(0 : x) \to (\mathbb{Q}/\mathbb{Z})^{++}$ be the linear map sending $[1]$ to the respective gluing such that in the first case $h([1]) = [\frac{1}{2}]$ and in the second case $h([1]) = [\frac{1}{d}]$. Assuming Zorn's lemma such that Proposition A.2(2) applies, there is an extension $\overline{h} : A \to (\mathbb{Q}/\mathbb{Z})^{++}$. This map is the desired map. If $\overline{h}(x) = h([1]) = 0$, then in the subtopos $d = 1$, hence $\nabla(x = 0)$. ☐

Proposition A.4. (1) *Every abelian group embeds canonically into a divisible abelian group.*
(2) *Every abelian group maps canonically to an abelian group functionally satisfying the Baer condition. Assuming Zorn's lemma, this map is an embedding.*

Proof. For the first part, an abelian group A embeds into the divisible group $\mathbb{Q}\langle A\rangle/K$, where $\mathbb{Q}\langle A\rangle$ is the underlying abelian group of the free $\mathbb{Q}$-vector space on A and K is the kernel of the canonical map $\mathbb{Z}\langle A\rangle \to A$.

The second part is more involved and requires, similar to the proof of Theorem 2.12, a strengthening of the Baer condition where the required extensions are explicitly given. As a first step, let A be an abelian group which is separated for a modal operator ∇ such that $\nabla(\varphi \vee (\varphi \Rightarrow \nabla\bot))$. The codomain of the canonical map

$$A \longrightarrow \prod_{g : A \to (\mathbb{Q}/\mathbb{Z})^{++}} (\mathbb{Q}/\mathbb{Z})^{++}, \; x \longmapsto (g(x))_g$$

is a product of abelian groups which functionally satisfy the Baer condition by Lemma A.3(1) and hence satisfies the Baer condition itself. Assuming Zorn's lemma, this map is injective by Lemma A.3(2) and ∇-separatedness of A.

For the general case, let an abelian group A be given. Denoting by $(\cdot)^{+_x}$ the plus construction with respect to the modal operator ∇_x with $\nabla_x\varphi :\equiv ((\varphi \Rightarrow x = 0) \Rightarrow x = 0)$, the codomain of the canonical map

$$A \longrightarrow \prod_{x \in A} \prod_{g : A \to (\mathbb{Q}/\mathbb{Z})^{+_x+_x}} (\mathbb{Q}/\mathbb{Z})^{+_x+_x}$$

satisfies the Baer condition and is injective as in the proof of Theorem 2.12. ☐

Proposition A.5. *Assuming Zorn's lemma:*

(1) *Every abelian group embeds canonically into an injective abelian group.*
(2) *Every module embeds canonically into an injective module.*
(3) *Every module in any Grothendieck topos embeds canonically into an injective module.*

Proof. The first claim is by Proposition A.4.

The second claim follows purely formally as in the proof of Proposition 2.6(3), using the adjunction between the category of modules and the category of abelian groups.

For the third claim, let $\mathcal{E}$ be a Grothendieck topos and let $\pi : \mathcal{F} \to \mathcal{E}$ be its Barr covering **XXX: ref**. Zorn's lemma passes from the metatheory to $\mathcal{F}$ since $\mathcal{F}$ is a localic topos. Hence the first part applies in $\mathcal{F}$. Purely formally, using the adjunction between $\mathrm{Ab}(\mathcal{F})$ and $\mathrm{Ab}(\mathcal{E})$, there are enough injective abelian groups in $\mathcal{E}$. The existence of enough injective modules in $\mathcal{E}$ follows from internalizing the second claim in $\mathcal{E}$. □

XXX: cite whom for the general argument?

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