Challenge. Without using Zorn, devise a spell for switching on Zorn: construct a Grothendieck topos with surjective geometric morphism to Set in which Zorn holds.

# Pointfreely embracing infinite sequences as useful fictions

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partially joint work with

   

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# What is a function in constructive mathematics?

Function types are brittle.

| | (naive) statement | cl. | sp. | in Eff | |
|---|---|---|---|---|---|
| 0 | Every function is computable. | ✗ | ? | ✓ | |
| 1 | Every function has a zero or not. | ✓ | ? | ✗ | |
| 2 | Every path leaves the Kleene tree. | ✗ | ? | ✓ | |
| 3 | There is no surjection $\mathbb{N} \twoheadrightarrow \mathbb{R}$. | ✓ | ? | ✓ | False in parametrized realizability. |

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With adverse effects throughout mathematics:

| | | | | | |
|---|---|---|---|---|---|
| 4 | Dickson's lemma. | ✓ | ? | ✗ | "Products of well quasi-orders are well." |
| 5 | Products of streamless sets are streamless. | ✓ | ? | ✗ | With LEM, true in Gr. toposes with enough points. |
| 6 | Products of wf. relations are wf. | ✓ | ? | | |
| 7 | Minimum principle for unif. cont. functions. | ✓ | ? | ✗ | "For $f : [0,1] \to \mathbb{R}_{>0}$, there is $c > 0$ with $f \geq c$." |
| 8 | Every field has an algebraic closure. | ✓ | ? | ✗ | Implies a form of choice. |
| 9 | Hilbert's basis theorem. | ✓ | ? | | "If $A$ is Noetherian, so is $A[X]$." |
| 10 | Locally Noetherian implies Noetherian. | ✓ | ? | | "If all $A[f_i^{-1}]$ are Noetherian, so is $A$." |
| 11 | Krull's lemma. | ✓ | ? | | "If $x \in \bigcap_{\mathfrak{p}} \mathfrak{p}$, then $x$ is nilpotent." |
| 12 | Fundamental theorem of Galois theory. | ✓ | ? | | "If $x \in L^{\mathrm{Aut}_K(L)}$, then $x \in K$." |

# Harnessing formal topology

Let $X$ be a set.

Irrespective of function types, there is a robust **formal space**[⋆] $X^{\mathbb{N}}$ with

- **basic opens:** $[x_0, \ldots, x_{n-1}]$, pictured as "$\{f : \mathbb{N} \to X \mid f(0) = x_0, \ldots, f(n-1) = x_{n-1}\}$"
- **coverings** inductively generated by: $[x_0, \ldots, x_{n-1}] \lhd \{[x_0, \ldots, x_{n-1}, a] \mid a \in X\}$

⋆ distributive lattice, site, locale, topos, arithmetic universe, …

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We can then explore:

$$\text{(absolute) points of } X^{\mathbb{N}} \quad \overset{1:1}{\longleftrightarrow} \quad \text{functions } \mathbb{N} \to X$$

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$$\begin{array}{rcl} \text{(absolute) points of } X^{\mathbb{N}} & \xleftrightarrow{1:1} & \text{functions } \mathbb{N} \to X \\ E\text{-valued points of } X^{\mathbb{N}} & \xleftrightarrow{1:1} & \text{morphisms } E \to X^{\mathbb{N}} \text{ of spaces} \end{array}$$

For suitable spaces $E$, $E$-valued points of $X^{\mathbb{N}}$ are for instance:

- Partially defined functions $f : \mathbb{N} \rightharpoonup X$ such that for all $n$, the value $f(n)$ is *not not* defined
- Total multivalued functions $f : \mathbb{N} \rightrightarrows X$ (approximately)

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# Dialogues with oracles

For a predicate $P$ on $\mathrm{List}(X)$, we inductively define $P \mid \sigma$ ("$P$ bars the list $\sigma$") by the following two clauses.

1. "now": If $P\,\sigma$, then $P \mid \sigma$.
2. "later": If $P \mid \sigma a$ for all $a \in X$, then $P \mid \sigma$.

The assertion $P \mid \sigma$ expresses in a positive way that **no matter how** the finite approximation $\sigma$ evolves over time to a **better approximation** $\tau$, **eventually** $P\,\tau$ will hold.

A witness of $P \mid \sigma$ is a **dialogue tree** recording **queries** to an oracle of type $\mathbb{N} \to X$, eventually culminating with witnesses of $P$ at the leaves.

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## A brittle definition

A set $X$ is streamless iff
for every function $f : \mathbb{N} \to X$,
there are numbers $i < j$ such that $f(i) = f(j)$.

Stability of streamless sets under products depends on foundational assumptions.

## A robust definition

A set $X$ is Noetherian iff $\mathrm{R} \mid [\,]$,
where $\mathrm{R}\,[x_0, \ldots, x_{n-1}]$ iff
there are numbers $i < j$ such that $x_i = x_j$.

Stability of Noetherian sets under products is provable in spartan mathematics.

# Elephant functions

**Def.** Let $X$ and $Y$ be spaces. An **elephant function** from $X$ to $Y$ consists of

- for each space $E$, a function $f_E : \mathrm{Pt}_E(X) \to \mathrm{Pt}_E(Y)$ between the sets of $E$-valued points

such that

- for each morphism $g : F \to E$, the diagram

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So a natural transformation $\text{よ}X \to \text{よ}Y$, so equivalently a morphism $X \to Y$ of spaces.

**Prop.** For a function $f : (\mathbb{N} \to \mathbb{2}) \to \mathbb{N}$, TFAE:

1. $f$ can be extended to an elephant function from $\mathbb{2}^{\mathbb{N}}$ to $\mathbb{N}$.
2. $f$ is uniformly continuous.
3. $f$ is eloquent (i.e. given by a dialogue).

**Prop.** For a function $f : [0, 1] \to \mathbb{R}$, TFAE:

1. $f$ can be extended to an elephant function from $[0, 1]$ to $\mathbb{R}$.
2. $f$ is uniformly continuous.
3. $f$ is eloquent (in a suitable real-number-oracle querying sense).

# Modal language for embracing points

**Def.** A statement $\varphi$ holds …

- **everywhere** ($\Box\varphi$) iff it holds **over every space**.
- **somewhere** ($\Diamond\varphi$) iff it holds **over some positive space**.
  (A space is *positive* iff every open covering of its top open is inhabited).
- **proximally** ($\diamonddiamond\varphi$) iff it holds **over some positive overt space**.

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for every function $f : \mathbb{N} \to X$,
there are numbers $i < j$ such that $f(i) = f(j)$.

**A robust definition**

A set $X$ is Noetherian iff $R \mid [\,]$,
where $R\,[x_0, \ldots, x_{n-1}]$ iff
there are numbers $i < j$ such that $x_i = x_j$.

**Equivalently:** $\Box(X \text{ is streamless})$.

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**Equivalently:** $\Box(X \text{ is streamless})$.

**A brittle definition**

A relation $R$ is well-founded$_\infty$ iff
for every decreasing sequence $f : \mathbb{N} \to X$, $\bot$.

**A robust definition**

A relation $R$ is well-founded$_{\text{ind}}$ iff $\mathrm{Acc}(R)$.

**Equivalently:** $\Box(R \text{ is well-founded}_\infty)$.

1. $(\Diamond\varphi) \iff \varphi$, if $\varphi$ is a geometric implication ("$\forall \ldots \forall(\% \Rightarrow \%)$", with no $\forall$ nor $\Rightarrow$ in $\%$).
2. $(\diamondsuit\varphi) \iff \varphi$, if $\varphi$ is a bounded first-order statement.

# Harmonized results

| | statement | cl. | sp. | in Eff |
|---|---|---|---|---|
| 0 | *Everywhere*, every function is computable. | ✗ | ✗ | ✗ |
| 1 | *Everywhere*, every function has a zero or not. | ✗ | ✗ | ✗ |
| 2 | *Everywhere*, every path leaves the Kleene tree. | ✗ | ✗ | ✗ |
| 3 | *Everywhere*, there is no surjection $\mathbb{N} \twoheadrightarrow \mathbb{R}$. | ✗ | ✗ | ✗ |
| 4 | Products of $\text{well}_{\text{ind}}$ quasi-orders are $\text{well}_{\text{ind}}$. | ✓ | ✓ | ✓ |
| 5 | Products of Noetherian sets are Noetherian. | ✓ | ✓ | ✓ |
| 6 | Products of $\text{well-founded}_{\text{ind}}$ relations are $\text{well-founded}_{\text{ind}}$. | ✓ | ✓ | ✓ |
| 7 | If *everywhere* $f(x) > 0$ for all $x$, then there is $c > 0$ with $f \geq c$. | ✓ | ✓ | ✓ |
| 8 | Every field has an algebraic closure *somewhere*. | ✓ | ✓ | ✓ |
| 9 | If $A$ is $\text{Noetherian}_{\text{ind}}$, so is $A[X]$. | ✓ | ✓ | ✓ |
| 10 | Locally $\text{Noetherian}_{\text{ind}}$ implies $\text{Noetherian}_{\text{ind}}$. | ✓ | ✓ | ✓ |
| 11 | If *everywhere* $x$ is included in every prime ideal, then $x$ is nilpotent. | ✓ | ✓ | ✓ |
| 12 | If *everywhere* $\sigma(x) = x$ holds for all $\sigma \in \mathrm{Aut}_K(L)$, then $x \in K$. | ✓ | ✓ | ✓ |
| 13 | For every inhabited set $X$, *proximally* there is a surjection $\mathbb{N} \twoheadrightarrow X$. | ✓ | ✓ | ✓ |
| 14 | *Somewhere*, LEM holds. | ✓ | ✓ | ✓ |
| 15 | Barr's theorem: Zorn $\Rightarrow \Box\Diamond$ AC. | ✓ | ✓ | ✓ |

# Taming the infinite

**Def.** Let $A$ be a ring.

- An infinite sequence $x_0, x_1, \ldots$ *stalls* iff there is a number $n$ with $x_n \in (x_0, \ldots, x_{n-1})$.
- A finite list $[x_0, \ldots, x_{m-1}]$ *stalls* iff there is a number $n < m$ with $x_n \in (x_0, \ldots, x_{n-1})$.
  Notation: "$\mathrm{Stalls}_A\, [x_0, \ldots, x_{m-1}]$"

**Def.** A ring $A$ is ...

- Noetherian$_\infty$ iff every infinite sequence of elements of $A$ stalls.
- Noetherian$_{\mathrm{ind}}$ iff $\mathrm{Stalls}_A \mid [\,]$.
  Equivalently: $\Box$($A$ is Noetherian$_\infty$).

**Thm. (classical)** Let $A$ be a Noetherian$_\infty$ ring. Then $A[X]$ is Noetherian$_\infty$ as well.

*Proof.* See [Hilbert 1890]. *This is not mathematics, it is theology!* □

**Thm.** Let $A$ be a Noetherian$_{\mathrm{ind}}$ ring. Then $A[X]$ is Noetherian$_{\mathrm{ind}}$ as well.

*Proof.* Let an arbitrary sequence of elements of $A[X]$ be given (over an arbitrary space). It suffices to verify that it stalls *somewhere*. *Somewhere,* LEM holds, and—crucially—*there* $A$ is still Noetherian$_\infty$. So *there* the sequence stalls by the classical proof. □

# Answering a question by Berardi–Buriola–Schuster

**Def.** Let $(X, \leq)$ be a quasi-order.

- An infinite sequence $x_0, x_1, \ldots$ is *good* iff there are numbers $i < j$ with $x_i \leq x_j$.
- A finite list $[x_0, \ldots, x_{m-1}]$ is *good* iff there are numbers $i < j < m$ with $x_i \leq x_j$.
  Notation: "Good $[x_0, \ldots, x_{m-1}]$"

**Def.** A quasi-order $X$ is …

- well$_\infty$ iff every infinite sequence is good.
- well$_{\text{ind}}$ iff Good $|$ $[\,]$ (equivalently: *everywhere*, $X$ is well$_\infty$).
- well$_{\text{impl}}$ iff for every monotone predicate $B$ of finite lists, $B \,|_{\text{incr}}\, [\,]$ implies $B \,|\, [\,]$ (equivalently: if, everywhere, every *increasing* infinite sequence $\alpha : \mathbb{N} \to X$ has a finite prefix validating $B$, then, everywhere, every infinite sequence does).

**Thm.** Let $X$ be a well$_{\text{ind}}$ quasi-order. Then $X$ is well$_{\text{impl}}$.

*Proof.* Let $B$ be a monotone predicate. Let $x_0, x_1, \ldots$ be an infinite sequence (over an arbitrary space). It suffices that it has a finite $B$-prefix *somewhere*. *Somewhere,* LEM holds. By a classical lemma, *there* we have an increasing infinite subsequence $x_{i_0}, x_{i_1}, \ldots$. By assumption, *there* we have a finite prefix with $B\,[x_{i_0}, \ldots, x_{i_n}]$. As $B$ is monotone, *there* we also have $B\,[x_0, x_1, \ldots, x_{i_n}]$. □

# Mismatches

Let $A$ be a ring.

1. Let $M$ be an $A$-module.

   The (robust) statement "*everywhere*, for every prime filter $\mathfrak{p}$ the stalk $M_\mathfrak{p}$ is finitely generated" is classically not equivalent (but stronger) than its single-worlded unmodal counterpart "for every absolute prime filter $\mathfrak{p}$, the stalk $M_\mathfrak{p}$ is finitely generated".

2. Let $f : A \to B$ be a homomorphism.

   The status of the (robust) statement "*everywhere*, the preimage of every maximal ideal of $B$ is maximal in $A$" is currently unclear.

   It might unfortunately be strictly stronger, classically, than "the preimage of every maximal ideal of $B$ is maximal in $A$".

   (Compare with Ryota Kuroki's pioneering work on the Nullstellensatz and Jacobson rings in constructive algebra.)

# Embracing implication as a useful fiction

(Bonus slide for people who are intrigued by arithmetic universes.)

Let $\mathbb{T}$ be a geometric theory. Let $U_{\mathbb{T}}$ be the generic $\mathbb{T}$-model in $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$.

Let $\mathbb{T}/U_{\mathbb{T}}$ be the theory (in $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$) of $U_{\mathbb{T}}$-algebras.

Let $A$ and $B$ be geometric formulas in the language of $\mathbb{T}/U_{\mathbb{T}}$ in a context $\vec{x}$. Then the assertion

$$\text{“}\forall \vec{x}.\, (A \Rightarrow B)\text{”}$$

makes use of implication and can therefore only be represented in $\mathrm{Sh}(\mathcal{C}_{\mathbb{T}})$ if Set supports implication. (It references the just one point of the classifying space of $\mathbb{T}/U_{\mathbb{T}}$, the initial one.)

In contrast, the assertion

$$\text{“}\mathbb{T}/U_{\mathbb{T}} \vdash (\forall \vec{x}.\, (A \Rightarrow B))\text{”}$$

is positive (a geometric formula) and hence available even if implication is not. (It can be understood as referencing the **generic point** of the classifying space of $\mathbb{T}/U_{\mathbb{T}}$.)

Curiously, these two assertions are equivalent (in a metatheory with implication).
*So use the latter as a proxy for the former!*