Embracing the generic prime ideal:
the tale of an enchanting mathematical phantom

– an invitation –

Antwerp Algebra Colloquium
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University of Augsburg

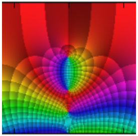
Mathematical phantoms

Gavin Wraith

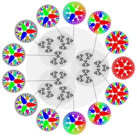
One of the recurring themes of mathematics, and one that I have always found seductive, is that of

- *the nonexistent entity which ought to be there but apparently is not;*
- *which nevertheless obtrudes its effects so convincingly that one is forced to concede a broader notion of existence.*

Examples



$\mathbb{C}$



$\mathbb{Q}_p$

$\mathbb{F}_1$

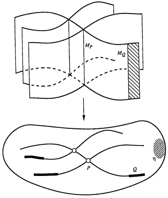
∞

The generic prime filter

Let A be a commutative ring with unit. Let M be an A-module. For any prime filter $\mathfrak{p} \subseteq A$, let

$$M_{\mathfrak{p}} := M[\mathfrak{p}^{-1}] := \{ \tfrac{x}{s} \mid x \in M, s \in \mathfrak{p} \}$$

be the **stalk** of M at $\mathfrak{p}$.



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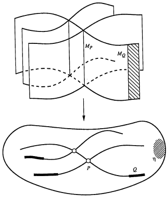

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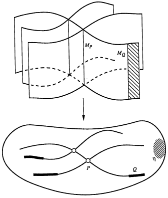

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Local-global principle.

$M = 0$	iff*	for all prime filters $\mathfrak{p}$, $M_{\mathfrak{p}} = 0$.
$M \to N$ is injective	iff*	for all prime filters $\mathfrak{p}$, $M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective.
$M \to N$ is surjective	iff*	for all prime filters $\mathfrak{p}$, $M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is surjective.
f is nilpotent in A	iff*	for all prime filters $\mathfrak{p}$, $f \not\in \mathfrak{p}$.
???	iff*	for all prime filters $\mathfrak{p}$, $M_{\mathfrak{p}}$ is fin. generated over $A_{\mathfrak{p}}$.
???	iff*	for all prime filters $\mathfrak{p}$, $M_{\mathfrak{p}}$ is finite free over $A_{\mathfrak{p}}$.

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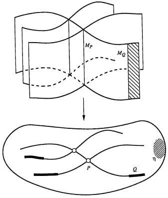

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> The generic prime filter is a **reification** of all prime filters into a single coherent entity.

Local-global principle. Let $\mathfrak{p}_0$ be the **generic prime filter** of A.

$M = 0$	iff	$M_{\mathfrak{p}_0} = 0$.
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$M \to N$ is surjective	iff	$M_{\mathfrak{p}_0} \to N_{\mathfrak{p}_0}$ is surjective.
f is nilpotent in A	iff	$f \notin \mathfrak{p}_0$.
M is fin. generated	iff	$M_{\mathfrak{p}_0}$ is fin. generated.
M is finite locally free	iff	$M_{\mathfrak{p}_0}$ is finite free.

The universal localization

Let A be a ring.

The stalks $A_{\mathfrak{p}}$ are **local rings**: If a finite sum of elements is invertible, then so is one of the summands.

Is there a **universal localization** of A?

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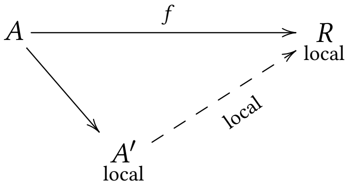

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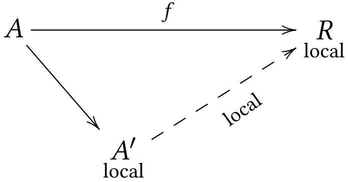

For a fixed local ring R, the localization $A' := A_\mathfrak{p}$ where $\mathfrak{p} := f^{-1}[R^\times]$ would do the job.

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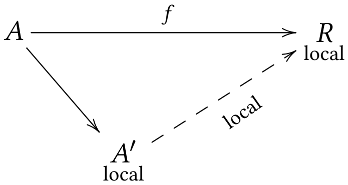

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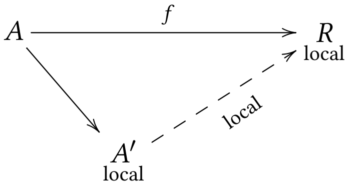

For a fixed local ring R, the localization $A' := A_\mathfrak{p}$ where $\mathfrak{p} := f^{-1}[R^\times]$ would do the job.

Fact. A universal localization exists iff A has exactly one prime filter.

Dream. If only there was a **generic prime filter** $\mathfrak{p}_0$, the universal localization would always exist and be given by $A^\sim := A_{\mathfrak{p}_0}$!

Generic models

Theorem. There is a **generic ring**, a particular ring* such that for every** ring-theoretic statement φ, the following are equivalent:

1. φ holds for the generic ring.
2. φ holds for every ring.
3. φ is provable from the ring axioms.

Similarly for every*** other theory in place of the theory of rings:

… the generic group, field, vector space, …

… the generic prime ideal of a given ring A …

… the generic surjection $\mathbb{N} \to X$ to a given set X …

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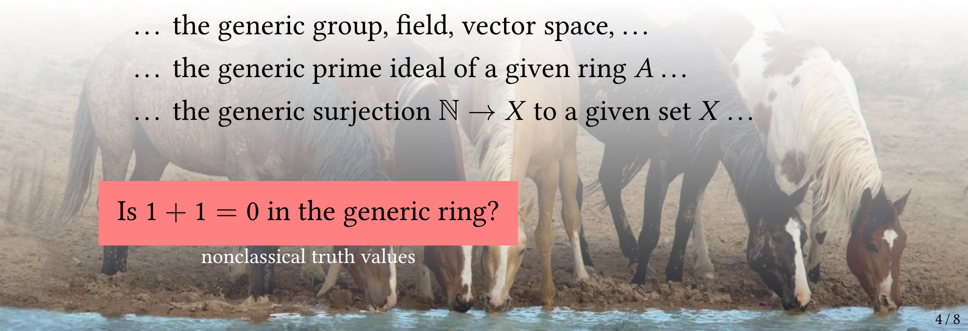
Is $1 + 1 = 0$ in the generic ring?

nonclassical truth values



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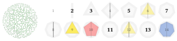
The generic ring is a field.

mysterious nongeometric sequents

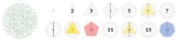
Mathematical universes

Set



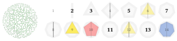
The usual laws of logic hold.

Sh(X)



The axiom of choice fails.

Eff



Every function is computable.

- For any topos $\mathcal{E}$ and any statement φ, we define the meaning of "$\mathcal{E} \models \varphi$" ("$\varphi$ holds in the internal universe of $\mathcal{E}$") using the **Kripke–Joyal semantics**.

- Any topos supports **mathematical reasoning**:

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Mathematical universes

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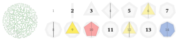

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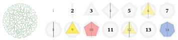

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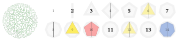

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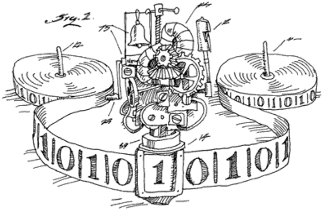
  no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

Exploring the effective topos



	Statement	in Set	in Eff
1	Every natural number is prime or not prime.	✓ (trivially)	✓
2	There are infinitely many primes.	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ is constantly zero or not.	✓ (trivially)	✗
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓

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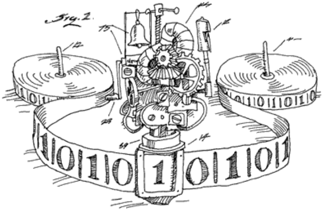

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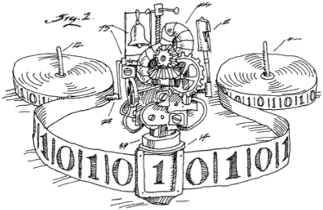
"Eff $\models$ 1" means: There is a machine which determines of any given number whether it is prime or not.

Exploring the effective topos



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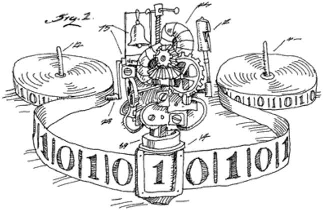
“Eff $\models$ 2” means: There is a machine producing arbitrarily many primes.

Exploring the effective topos



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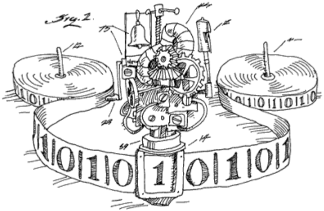
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓

"Eff $\models$ 3" means: There is a machine which, given a machine computing a map $f : \mathbb{N} \to \mathbb{N}$, determines whether f is constantly zero or not.

Exploring the effective topos



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"Eff $\models$ 4" means: There is a machine which, given a machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs a machine computing f.

The classifying topos as a local lens

- For ring elements $f \in A$ and formulas φ, we define $D(f) \models \varphi$ ("φ holds on (and beyond) stage $D(f)$") by the following clauses.
- A formula φ holds in the classifying topos of the theory of prime filters of A iff $D(1) \models \varphi$.

$D(f) \models \forall x : A^\sim .\, \varphi(x)$	iff for all $g \in A$ and $x_0 \in A[(fg)^{-1}]$, $D(fg) \models \varphi(x_0)$
$D(f) \models \varphi \Rightarrow \psi$	iff for all $g \in A$, $D(fg) \models \varphi$ implies $D(fg) \models \psi$
$D(f) \models \varphi \vee \psi$	iff there is a partition $f^n = fg_1 + \cdots + fg_m$ s. th. for each i, $D(fg_i) \models \varphi$ or $D(fg_i) \models \psi$
$D(f) \models \bot$	iff f is nilpotent
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Example.

$D(1) \models$ 'x is not invertible' iff $D(1) \models$ 'x is invertible' $\Rightarrow \bot$

iff for all $g \in A$, if $D(g) \models$ 'x is invertible' then $D(g) \models \bot$

iff for all $g \in A$, if x is invertible in $A[g^{-1}]$ then g is nilpotent iff x is nilpotent.

Applications of the generic prime filter

Injective matrices

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Proof. Assume not. Then there is a minimal prime ideal $\mathfrak{p} \subseteq A$. The matrix is injective over the field $A_{\mathfrak{p}}$; contradiction to basic linear algebra. □

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Proof. 'M is also injective as a matrix over $A^{\sim} = A_{\mathfrak{p}_0}$. This is a contradiction by basic intuitionistic linear algebra.' Thus '$\perp$'. Hence $1 = 0$ in A. □

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Grothendieck's generic freeness

Theorem. Let M be a finitely generated A-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is a free $A[f^{-1}]$-module, then $1 = 0$ in A.

Proof. The claim amounts to '$M^\sim$ is **not not** free'. This statement follows from basic intuitionistic linear algebra over the field $A^\sim$. □