A general Nullstellensatz for generalised spaces

– an invitation –

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Fig.: The Event Horizon Telescope picture of the central black hole in the galaxy M87

The mystery of nongeometric sequents

Let $\mathbb{T}$ be a geometric theory, for instance the theory of rings.

sorts, function symbols, relation symbols, geometric sequents as axioms

sorts: R
fun. symb.: 0, 1, $-$, $+$, $\cdot$
axioms: $(\top \vdash_{x,y:R} xy = yx)$, …

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Theorem. There is a **generic model** $U_{\mathbb{T}}$. It is **conservative** in that for any geometric sequent σ the following notions coincide:

1. The sequent σ holds for $U_{\mathbb{T}}$.
2. The sequent σ holds for any $\mathbb{T}$-model in any topos.
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Observation (Kock). The generic local ring is a field:

$$(x = 0 \Rightarrow \bot) \vdash_{x:R} (\exists y : R.\ xy = 1)$$

Construction of the generic model

The generic model is **not** the same as …

- the **initial model** (think $\mathbb{Z}$) or
- the **free model on one generator** (think $\mathbb{Z}[X]$).

Set-based models are **too inflexible**.

Definition. The **syntactic site** $\mathcal{C}_{\mathbb{T}}$ has …

1. objects: $\{x_1 : X_1, \ldots, x_n : X_n.\, \varphi\}$ (shorter: $\{\vec{x}.\, \varphi\}$)
2. morphisms: eqv. classes of provably functional formulas
3. coverings: provably jointly surjective families

The topos of sheaves over $\mathcal{C}_{\mathbb{T}}$ is the **classifying topos** $\mathrm{Set}[\mathbb{T}]$.
The generic model interprets a sort X by $よ\{x : X.\, \top\}$.

Working internally to toposes

Let $\mathcal{C}$ be a site. We recursively define

$$U \models \varphi \quad (\text{“}\varphi\text{ holds on }U\text{”})$$

for objects $U \in \mathcal{C}$ and formulas φ. Write “$\mathrm{Sh}(\mathcal{C}) \models \varphi$” for $1 \models \varphi$.

$U \models \top$ iff true

$U \models \bot$ iff ~~false~~ the empty family is a covering of U

$U \models s = t : F$ iff $s|_U = t|_U \in F(U)$

$U \models \varphi \wedge \psi$ iff $U \models \varphi$ and $U \models \psi$

$U \models \varphi \vee \psi$ iff ~~$U \models \varphi$ or $U \models \psi$~~ there exists a covering $(U_i \to U)_i$
such that for all i: $U_i \models \varphi$ or $U_i \models \psi$

$U \models \varphi \Rightarrow \psi$ iff for all $V \to U$: $V \models \varphi$ implies $V \models \psi$

$U \models \forall s : F.\, \varphi(s)$ iff for all $V \to U$ and sections $s_0 \in F(V)$: $V \models \varphi(s_0)$

$U \models \exists s : F.\, \varphi(s)$ iff ~~there exists $s_0 \in F(U)$ such that $U \models \varphi(s_0)$~~
there exists a covering $(U_i \to U)_i$ such that for all i:
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A selection of nongeometric properties

The **generic object** validates:

1. $\forall x, y : U_{\mathbb{T}}.\ \neg\neg(x = y)$.
2. $\forall x_1, \ldots, x_n : U_{\mathbb{T}}.\ \neg\forall y : U_{\mathbb{T}}.\ \bigvee_{i=1}^{n} y = x_i$.
3. $(U_{\mathbb{T}})^{U_{\mathbb{T}}} \cong 1 \amalg U_{\mathbb{T}}$.

The **generic ring** validates:

1. $\forall x : U_{\mathbb{T}}.\ \neg\neg(x = 0)$.
2. $\forall x : U_{\mathbb{T}}.\ (x = 0 \Rightarrow 1 = 0) \Rightarrow (\exists y : U_{\mathbb{T}}.\ xy = 1)$.

The **generic local ring** validates:

1. $\neg\forall x : U_{\mathbb{T}}.\ \neg\neg(x = 0)$.
2. $\forall a_0, \ldots, a_{n-1} : U_{\mathbb{T}}.\ \neg\neg\exists x : U_{\mathbb{T}}.\ x^n + a_{n-1}x^{n-1} + \cdots + a_0 x^0 = 0$.
3. Let $\Delta = \{\varepsilon : U_{\mathbb{T}} \mid \varepsilon^2 = 0\}$. For any map $f : \Delta \to U_{\mathbb{T}}$, there are unique elements $a, b : U_{\mathbb{T}}$ s. th. $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon : \Delta$.

Affine schemes

Let A be a ring. Is there a **free local ring** $A \to A'$ over A?

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] & & \underset{\text{local}}{R} \\
& \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

For a fixed ring R, the localization $A' := A[S^{-1}]$ with $S := f^{-1}[R^\times]$ would do the job. (S is a *filter*.)

Hence we need the **generic filter**.

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If A is reduced ($x^n = 0 \Rightarrow x = 0$):

$A^{\sim}$ is a **field**: $\forall x : A^{\sim}.\, (\neg(\exists y : A^{\sim}.\, xy = 1) \Rightarrow x = 0)$.
$A^{\sim}$ has **¬¬-stable equality**: $\forall x, y : A^{\sim}.\, \neg\neg(x = y) \Rightarrow x = y$.
$A^{\sim}$ is **anonymously Noetherian**.

A systematic source of nongeometricity?

Empirical fact. In **synthetic algebraic geometry**, every known property of $\underline{\mathbb{A}}^1$ followed from its **synthetic quasicoherence**:

> *For any finitely presented $\underline{\mathbb{A}}^1$-algebra A, the canonical map*
>
> $$A \longrightarrow (\underline{\mathbb{A}}^1)^{\mathrm{Hom}_{\underline{\mathbb{A}}^1}(A,\underline{\mathbb{A}}^1)}, \ s \longmapsto (x \mapsto x(s))$$
>
> *is an isomorphism of $\underline{\mathbb{A}}^1$-algebras.*

1. Does a general metatheorem explain this observation?
2. Is there a systematic source in any classifying topos?
3. Is there even an exhaustive source?

$$\begin{tikzcd}
\mathbb{T}\text{ proves }\alpha \arrow[r, Rightarrow, bend right] & \alpha\text{ holds for }U_{\mathbb{T}} \arrow[l, Rightarrow, bend right, "\times"'] \arrow[r, Rightarrow, bend right] & \alpha\text{ is }\mathbb{T}\text{-redundant} \arrow[l, Rightarrow, bend right, "\times"']
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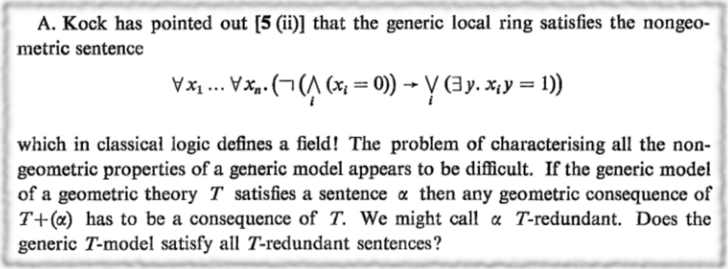

A. Kock has pointed out [5 (ii)] that the generic local ring satisfies the nongeometric sentence

$$\forall x_1 \ldots \forall x_n . (\neg (\bigwedge_i (x_i = 0)) \rightarrow \bigvee_i (\exists y . x_i y = 1))$$

which in classical logic defines a field! The problem of characterising all the nongeometric properties of a generic model appears to be difficult. If the generic model of a geometric theory T satisfies a sentence α then any geometric consequence of $T+(\alpha)$ has to be a consequence of T. We might call α T-redundant. Does the generic T-model satisfy all T-redundant sentences?

Gavin Wraith. Some recent developments in topos theory.
In: Proc. of the ICM (Helsinki, 1978).

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A topos-theoretic Nullstellensatz

Theorem. Internally to $\mathrm{Set}[\mathbb{T}]$:

For any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.

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The algebraic Nullstellensatz. Let A be a ring. Let $f, g \in A[X]$ be polynomials. Then, subject to some conditions:

$$\underbrace{\big(\forall x \in A.\, (f(x) = 0 \Rightarrow g(x) = 0)\big)}_{\text{algebraic truth}} \Longrightarrow \underbrace{\big(\exists h \in A[X].\, g = hf\big)}_{\text{algebraic certificate}}$$

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A naive version. "Internally to $\mathrm{Set}[\mathbb{T}]$, for any geometric sequent σ over the signature of $\underline{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}$ proves σ." **False,** for instance with the theory of rings we have

$$\begin{aligned} &\mathrm{Set}[\mathbb{T}] \models \neg(\ulcorner \underline{\mathbb{T}} \text{ proves } (\top \vdash 1 + 1 = 0)\urcorner) \\ \text{but} \quad &\mathrm{Set}[\mathbb{T}] \not\models \neg(1 + 1 = 0). \end{aligned}$$

A varying internal theory

Theorem. Internally to $\mathrm{Set}[\mathbb{T}]$:

For any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.

Definition. The theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is the internal geometric theory of **$U_{\mathbb{T}}$-algebras**, the theory which arises from $\underline{\mathbb{T}}$ by adding:

1. for each element $x : U_{\mathbb{T}}$ a constant symbol e_x,
2. for each function symbol f and n-tuple $(x_1, \ldots, x_n) \in (U_{\mathbb{T}})^n$ the axiom $(\top \vdash f(e_{x_1}, \ldots, e_{x_n}) = e_{f(x_1,\ldots,x_n)})$,
3. for each relation symbol R and n-tuple $(x_1, \ldots, x_n) \in (U_{\mathbb{T}})^n$ such that $R(x_1, \ldots, x_n)$ the axiom $(\top \vdash R(e_{x_1}, \ldots, e_{x_n}))$.

Remark. Externalising the internal classifying topos $\mathrm{Set}[\mathbb{T}][\underline{\mathbb{T}}/U_{\mathbb{T}}]$ yields the classifying topos of $\mathbb{T}$-homomorphisms.

Revisiting the test cases

Theorem. Internally to $\mathrm{Set}[\mathbb{T}]$:

> *For any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.*

In the object classifier. Let $x, y : U_{\mathbb{T}}$. Assume that $\neg(x = y)$. By the Nullstellensatz $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(e_x = e_y \vdash \bot)$. But this is false in the $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model $U_{\mathbb{T}}/(x \sim y)$.

In the ring classifier. Let $f, g : U_{\mathbb{T}}[X]$ such that any zero of f is a zero of g. By the Nullstellensatz $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves this fact. Hence it holds in the $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model $U_{\mathbb{T}}[X]/(f)$. In this model f has the zero $[X]$. Hence also $g([X]) = 0$ in $U_{\mathbb{T}}[X]/(f)$, that is $g = hf$ for some $h : U_{\mathbb{T}}[X]$.

Exhaustion and extensions

Theorem. A first-order formula holds for $U_{\mathbb{T}}$ iff it is intuitionistically provable from the axioms of $\mathbb{T}$ and the scheme

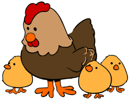
$$\ulcorner \sigma \text{ holds} \urcorner \implies \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves}^{\star\star}\ \sigma \urcorner. \qquad \text{(Nullstellensatz)}$$

Theorem. Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$. Assume that $U_{\mathbb{T}}$ is contained in the subtopos $\mathrm{Set}[\mathbb{T}']$. Then internally to $\mathrm{Set}[\mathbb{T}']$:

> *A geometric* sequent σ with Horn consequent holds for $U_{\mathbb{T}'}$ iff $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.*

Theorem. The morphism ev is an isomorphism.

$$\mathrm{ev} : \mathrm{FunctFormulas}^\star(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash) \longrightarrow P(U_{\mathbb{T}})$$



Theorem. A higher-order formula holds for $U_{\mathbb{T}}$ iff it is provable in intuitionistic higher-order logic from the axioms of $\mathbb{T}$ and the higher-order Nullstellensatz scheme.