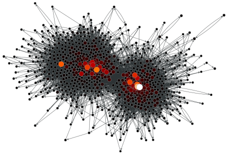
Maximal ideals in commutative algebra as convenient fictions

– an invitation –

Bonn Constructive Algebra Seminar
July 22th, 2024



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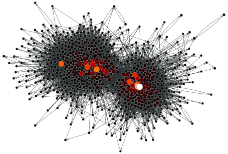
$$\begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

Then for every parameter value $t \in \Omega$, classically there is

- a full list of eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ and
- an eigenvector basis $(v_1(t), \ldots, v_n(t))$.

Can locally the functions λ_i be chosen to be continuous?
How about the v_i?



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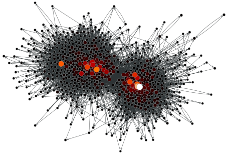
How about the v_i? **No**



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Can locally the functions λ_i be chosen to be continuous? **Yes.**
How about the v_i? **No**, but **yes** on a dense open subset of Ω.

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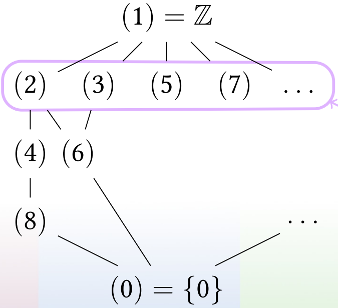

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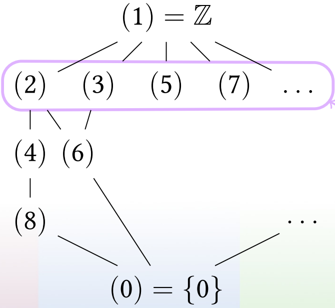

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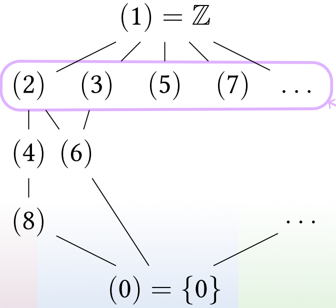

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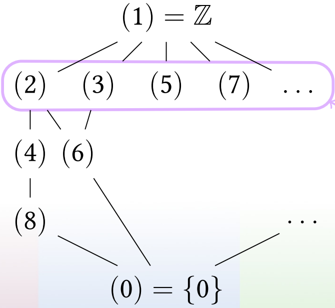

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2. Yes, if A is countable and membership of finitely generated ideals is decidable:
   Let $A = \{x_0, x_1, \ldots\}$. Then set:

$$\mathfrak{m}_0 := \{0\}, \qquad \mathfrak{m}_{n+1} := \begin{cases} \mathfrak{m}_n + (x_n), & \text{if } 1 \not\in \mathfrak{m}_n + (x_n), \\ \mathfrak{m}_n, & \text{else.} \end{cases}$$

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"a bad joke"

"non-informative"

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4. In the general case: No, but yes in a *suitable forcing extension*, and *bounded first-order consequences* of its existence there do hold in the base universe.

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Proof (constructive, special case). Write $M = \binom{x}{y}$. By surjectivity, we have $u, v \in A$ with

$$u \binom{x}{y} = \binom{1}{0} \quad \text{and} \quad v \binom{x}{y} = \binom{0}{1}.$$

Hence $1 = (vy)(ux) = (uy)(vx) = 0$. □

Abstract proofs should be blueprints for concrete ones.

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Proof. (offensive?) By LEM, there is a minimum $\alpha(i)$. Set $j := i + 1$. $\square$

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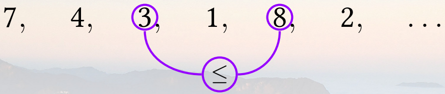

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Examples. $(\mathbb{N}, \leq)$, $X \times Y$, X^*, $\mathrm{Tree}(X)$.

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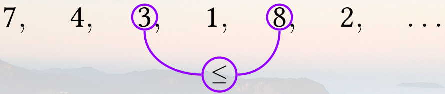

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Def. A preorder is **well** iff any of the following equivalent conditions hold:

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1. The **generic sequence** $\mathbb{N} \to X$ is good.
2. Every sequence $\mathbb{N} \to X$ **in every forcing extension** is good.
3. There is a **well-founded tree** witnessing universal goodness.

Ingredients for forcing

To construct a forcing extension, we require:

1. a base universe V
2. a preorder L of **forcing conditions** in V, pictured as **finite approximations** (*convention:* $\tau \preccurlyeq \sigma$ means that τ is a better finite approximation than σ)
3. a **covering system** governing how finite approximations evolve to better ones (for each $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$, with a simulation condition)

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Use **finite lists** $\sigma \in X^*$ as forcing conditions, where $\tau \preccurlyeq \sigma$ iff σ is an initial segment of τ, and be prepared to grow σ to ...

(a) one of $\{\sigma x \mid x \in X\}$, to make σ more defined

(b) one of $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$, for any $a \in X$, to make σ more surjective

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For the generic prime ideal of a ring A

Use **f.g. ideals** as forcing conditions, where $\mathfrak{b} \preccurlyeq \mathfrak{a}$ iff $\mathfrak{b} \supseteq \mathfrak{a}$, and be prepared to grow $\mathfrak{a}$ to …

(a) one of $\emptyset$, if $1 \in \mathfrak{a}$, to make $\mathfrak{a}$ more proper

(b) one of $\{\mathfrak{a} + (x), \mathfrak{a} + (y)\}$, if $xy \in \mathfrak{a}$, to make $\mathfrak{a}$ more prime

The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

- Repeats $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i = x_j$
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We then define "$P \mid \sigma$" ("P bars σ") inductively by the following clauses:

1. If $P\sigma$, then $P \mid \sigma$.
2. If $P \mid \tau$ for all $\tau \in R$, where R is some covering of σ, then $P \mid \sigma$.

So $P \mid \sigma$ expresses in a **direct inductive fashion**:

"No matter how σ evolves to a better approximation τ, eventually $P\tau$ will hold."

The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

- Repeats $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i = x_j$
- Good $x_0 \ldots x_{n-1} :\equiv \exists i.\, \exists j.\, i < j \wedge x_i \leq x_j$ (for some preorder $\leq$ on X)

We then define "$P \mid \sigma$" ("P bars σ") inductively by the following clauses:

1. If $P\sigma$, then $P \mid \sigma$.
2. If $P \mid \tau$ for all $\tau \in R$, where R is some covering of σ, then $P \mid \sigma$.

So $P \mid \sigma$ expresses in a **direct inductive fashion**:

> "No matter how σ evolves to a better approximation τ, eventually $P\tau$ will hold."

We use quantifier-like notation: "$\nabla(\tau \preccurlyeq \sigma).\, P\tau$" means "$P \mid \sigma$".

Proof translations

Thm. Every IQC-proof remains correct, with at most a polynomial increase in length, if throughout we replace

$$\begin{array}{ccccrcl} \exists & \rightsquigarrow & \exists^{\text{cl}}, & \text{where} & \exists^{\text{cl}} & :\equiv & \neg\neg\exists, \\ \vee & \rightsquigarrow & \vee^{\text{cl}}, & \text{where} & \alpha \vee^{\text{cl}} \beta & :\equiv & \neg\neg(\alpha \vee \beta), \\ = & \rightsquigarrow & =^{\text{cl}}, & \text{where} & s =^{\text{cl}} t & :\equiv & \neg\neg(s = t). \end{array}$$

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When we say: some statement “holds in $V^{\neg\neg}$”,
we mean: its translation holds in V.

Similarly for arbitrary forcing extensions V^{∇}, “just with ∇ instead of $\neg\neg$”.

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Similarly for arbitrary forcing extensions V^{∇}, "just with ∇ instead of $\neg\neg$".

Ex. As $\neg\neg(\varphi \vee \neg\varphi)$ is a theorem of IQC, the law of excluded middle holds in $V^{\neg\neg}$.

The ∇-translation

For bounded first-order formulas over the (large) first-order signature which has

1. one sort $\underline{X}$ for each set X in the base universe,
2. one n-ary function symbol $\underline{f} : \underline{X_1} \times \cdots \times \underline{X_n} \to \underline{Y}$ for each map $f : X_1 \times \cdots \times X_n \to Y$,
3. one n-ary relation symbol $\underline{R} \hookrightarrow \underline{X_1} \times \cdots \times \underline{X_n}$ for each relation $R \subseteq X_1 \times \cdots \times X_n$, and
4. an additional unary relation symbol $G \hookrightarrow \underline{L}$ (for the *generic filter* of L),

we recursively define:

$\sigma \models s = t$	iff	$\nabla\sigma.\, [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\, (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\, \sigma \preccurlyeq [\![\tau]\!]$.
$\sigma \models \top$	iff	$\top$.	$\sigma \models \bot$	iff	$\nabla\sigma.\, \bot$
$\sigma \models \varphi \wedge \psi$	iff	$(\sigma \models \varphi) \wedge (\sigma \models \psi)$.	$\sigma \models \varphi \vee \psi$	iff	$\nabla\sigma.\, (\sigma \models \varphi) \vee (\sigma \models \psi)$.
$\sigma \models \forall(x : \underline{X}).\, \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\, \forall(x_0 \in X).\, \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\, \varphi$	iff	$\nabla\sigma.\, \exists(x_0 \in X).\, \sigma \models \varphi[\underline{x_0}/x]$.

Finally, we say that φ "holds in V^∇" iff for all $\sigma \in L$, $\sigma \models \varphi$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma \in X^*).\, \forall(a \in X).\, \nabla(\tau \preccurlyeq \sigma).\, \exists(n \in \mathbb{N}).\, \tau[n] = a$.

The ∇-translation

$\sigma \models s = t$ iff $\nabla\sigma.\ [\![s]\!] = [\![t]\!]$.

$\sigma \models \varphi \Rightarrow \psi$ iff $\forall(\tau \preccurlyeq \sigma).\ (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.

$\sigma \models \top$ iff $\top$.

$\sigma \models \varphi \wedge \psi$ iff $(\sigma \models \varphi) \wedge (\sigma \models \psi)$.

$\sigma \models \forall(x : \underline{X}).\ \varphi$ iff $\forall(\tau \preccurlyeq \sigma).\ \forall(x_0 \in X).\ \tau \models \varphi[\underline{x_0}/x]$.

$\sigma \models \underline{R}(s_1, \ldots, s_n)$ iff $\nabla\sigma.\ R([\![s_1]\!], \ldots, [\![s_n]\!])$.

$\sigma \models G\tau$ iff $\nabla\sigma.\ \sigma \preccurlyeq [\![\tau]\!]$.

$\sigma \models \bot$ iff $\nabla\sigma.\ \bot$

$\sigma \models \varphi \vee \psi$ iff $\nabla\sigma.\ (\sigma \models \varphi) \vee (\sigma \models \psi)$.

$\sigma \models \exists(x : \underline{X}).\ \varphi$ iff $\nabla\sigma.\ \exists(x_0 \in X).\ \sigma \models \varphi[\underline{x_0}/x]$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma \in X^*).\ \forall(a \in X).\ \nabla(\tau \preccurlyeq \sigma).\ \exists(n \in \mathbb{N}).\ \tau[n] = a.$
map $\mathbb{N} \to X$	"the gen. sequence is good"	Good | [ ].
frame of opens	"every complex number has a square root"	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.

The ∇-translation

$\sigma \models s = t$	iff	$\nabla\sigma.\ [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\ R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\ (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\ \sigma \preccurlyeq [\![\tau]\!]$.
$\sigma \models \top$	iff	$\top$.	$\sigma \models \bot$	iff	$\nabla\sigma.\ \bot$
$\sigma \models \varphi \wedge \psi$	iff	$(\sigma \models \varphi) \wedge (\sigma \models \psi)$.	$\sigma \models \varphi \vee \psi$	iff	$\nabla\sigma.\ (\sigma \models \varphi) \vee (\sigma \models \psi)$.
$\sigma \models \forall(x : \underline{X}).\ \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\ \forall(x_0 \in X).\ \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\ \varphi$	iff	$\nabla\sigma.\ \exists(x_0 \in X).\ \sigma \models \varphi[\underline{x_0}/x]$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(\sigma \in X^*).\ \forall(a \in X).\ \nabla(\tau \preccurlyeq \sigma).\ \exists(n \in \mathbb{N}).\ \tau[n] = a$.
map $\mathbb{N} \to X$	"the gen. sequence is good"	Good $\mid$ [ ].
frame of opens	"every complex number has a square root"	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.
little Zariski	"every f.g. vector space does *not not* have a basis"	**Grothendieck's generic freeness lemma**

Outlook

Passing to and from extensions

Thm. Let φ be a **bounded first-order formula** not mentioning G. In each of the following situations, we have that φ holds in V^∇ iff φ holds in V:

1. L and all coverings are inhabited (proximality).
2. L contains a top element, every covering of the top element is inhabited, and φ is a coherent implication (positivity).

The mystery of nongeometric sequents

The **generic ideal** of a ring is maximal:

$$(x \in \mathfrak{a} \Rightarrow 1 \in \mathfrak{a}) \Longrightarrow 1 \in \mathfrak{a} + (x).$$

The **generic ring** is a field:

$$(x = 0 \Rightarrow 1 = 0) \Longrightarrow (\exists y.\, xy = 1).$$

Traveling the multiverse …

LEM is a **switch** and **holds positively**; being countable is a **button**.

Every instance of DC **holds proximally**.

A geometric implication is provable iff it holds **everywhere**.

… upwards, but always keeping ties to the base.

Formalities

Def. A **forcing notion** consists of a preorder L of **forcing conditions**, and for every $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$ of **coverings** of σ such that: If $\tau \preccurlyeq \sigma$ and $R \in \mathrm{Cov}(\sigma)$, there should be a covering $S \in \mathrm{Cov}(\tau)$ such that $S \subseteq \downarrow R$.

	preorder L	coverings of an element $\sigma \in L$	filters of L
1	X^*	$\{\sigma x \mid x \in X\}$	maps $\mathbb{N} \to X$
2	X^*	$\{\sigma x \mid x \in X\}$, $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$ for each $a \in X$	surjections $\mathbb{N} \twoheadrightarrow X$
3	f.g. ideals	—	ideals
4	f.g. ideals	$\{\sigma + (a), \sigma + (b)\}$ for each $ab \in \sigma$, $\{\}$ if $1 \in \sigma$	prime ideals
5	opens	$\mathcal{U}$ such that $\sigma = \bigcup \mathcal{U}$	points
6	$\{\star\}$	$\{\star \mid \varphi\} \cup \{\star \mid \neg\varphi\}$	witnesses of LEM

Def. A *filter* of a forcing notion (L, Cov) is a subset $F \subseteq L$ such that

1. F is upward-closed: if $\tau \preccurlyeq \sigma$ and if $\tau \in F$, then $\sigma \in F$;
2. F is downward-directed: F is inhabited, and if $\alpha, \beta \in F$, then there is a common refinement $\sigma \preccurlyeq \alpha, \beta$ such that $\sigma \in F$; and
3. F splits the covering system: if $\sigma \in F$ and $R \in \mathrm{Cov}(\sigma)$, then $\tau \in F$ for some $\tau \in R$.