Revisiting **divisible**, **injective** and **flabby** abelian groups from a **constructive point of view**

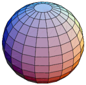
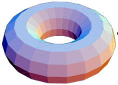
– an invitation –

REDCOM:
Reducing complexity in algebra, logic, combinatorics

Brixen
September 21th, 2022

Ingo Blechschmidt
University of Augsburg

Cohomology

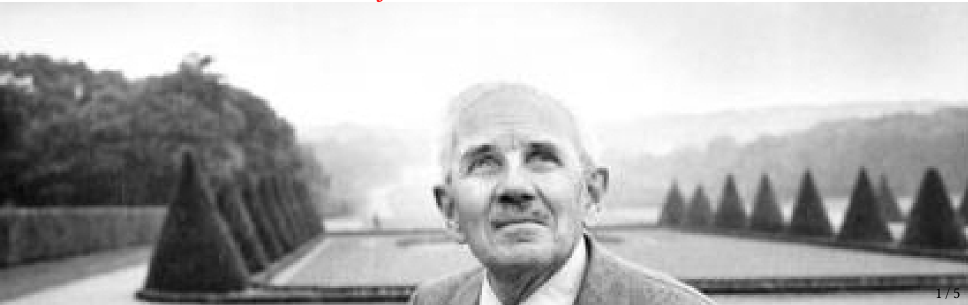
Is  homeomorphic to ? No:

$H^0($$, \mathbb{Z}) \cong \mathbb{Z}$ $H^1($$, \mathbb{Z}) \cong 0$ $H^2($$, \mathbb{Z}) \cong \mathbb{Z}$

$H^0($$, \mathbb{Z}) \cong \mathbb{Z}$ $H^1($$, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ $H^2($$, \mathbb{Z}) \cong \mathbb{Z}$

Basic tool for sheaf cohomology and derived functors:

injective resolutions



Integrated development?

Prop. There are **irrational** numbers x, y with x^y **rational**.

Proof. Either $\sqrt{2}^{\sqrt{2}}$ is rational or not. In the first case we are done. In the second case we set $x := \sqrt{2}^{\sqrt{2}}$ and $y := \sqrt{2}$. $\square$

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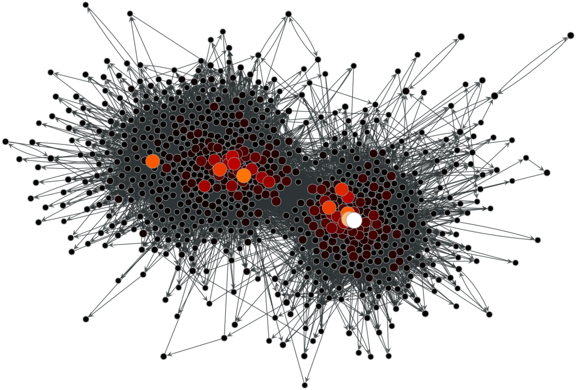
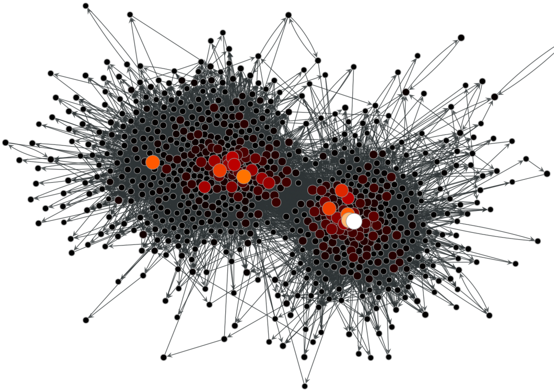
1. **Constructively,** we work without LEM, AC and ZORN.
2. Constructive proofs always yield **algorithms** and **global versions** for continuous families.
3. **Concrete** classical results can **always be constructivized**: If ZFC proves an arithmetical Π^0_2-statement, so does IZF.



local “Every real symmetric matrix does have an eigenvector.” ✓

global “For every continuous family of symmetric matrices, eigenvectors can locally be picked continuously.” ?



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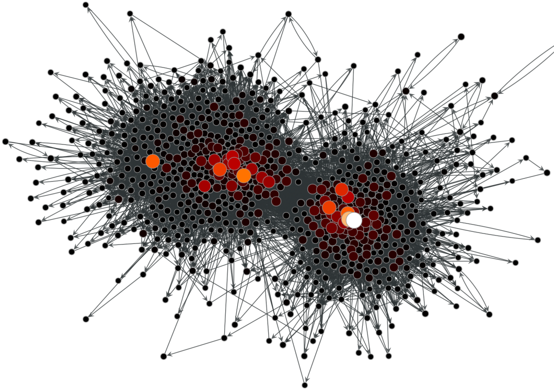
“Let X be a topological space and let $A : X \to M_n^{\mathrm{sym}}(\mathbb{R})$ be a continuous map to the space of symmetric $(n \times n)$-matrices. Then there is an open covering $\bigcup_{i \in I} U_i$ of X such that or all indices $i \in I$, there is a continuous map $v : U_i \to \mathbb{R}^n$ such that for all $x \in U_i$, the vector $v(x)$ is an eigenvector of $A(x)$.”



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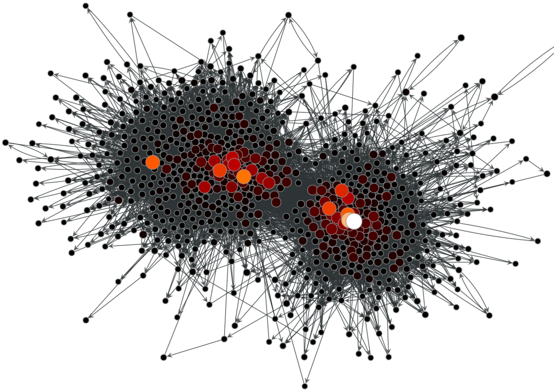

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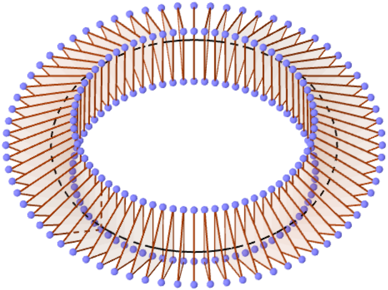

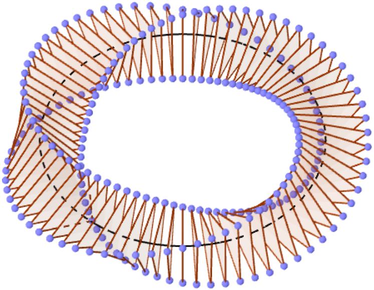

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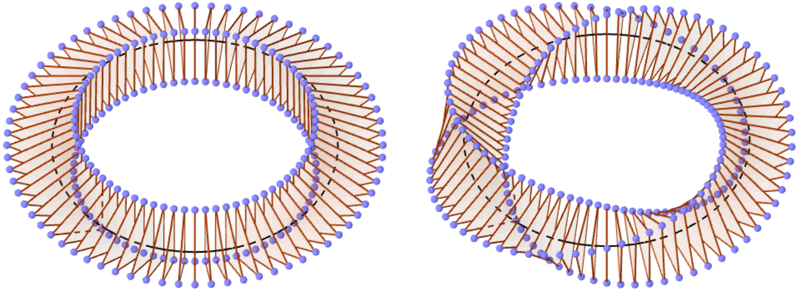

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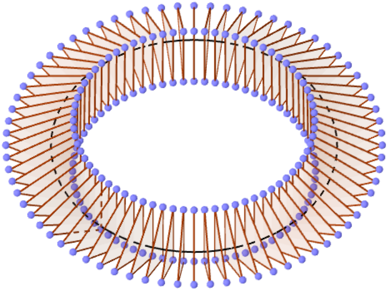 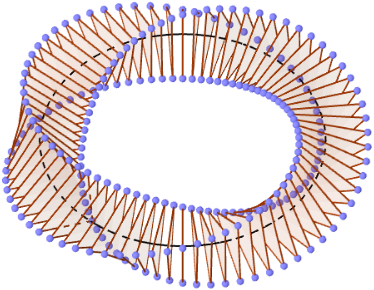

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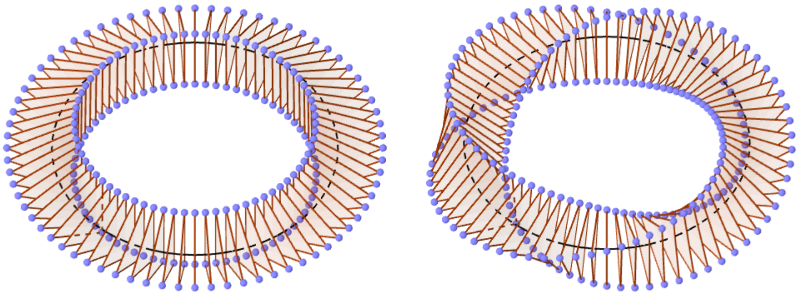

local “Let M be a finitely generated module over a field k.
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“Let M be a finitely generated module over an arbitrary commutative ring A. **If $f = 0$ is the only element of A** such that $M[f^{-1}]$ is finite free over $A[f^{-1}]$, **then $1 = 0$ in A**.”

local "Let R be a ring. Let $n \geq 0$ be an integer. We have

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \textit{(omitted)}$$

as R-modules." ✓

global "Let S be a scheme. Let $n \geq 1$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$. For the structure morphism $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S$, we have

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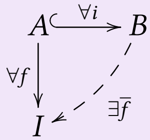

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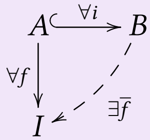

Def. A group I is **divisible** iff for every element $x \in I$ and every $n \in \mathbb{N}_{>0}$, there is an element $y \in I$ such that $ny = x$.

Examples. $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}(p^\infty)$.

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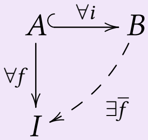

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   Every divisible group is injective.
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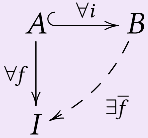

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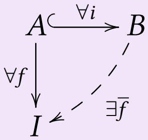

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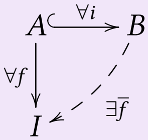

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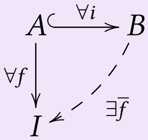

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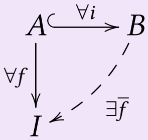

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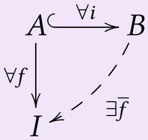

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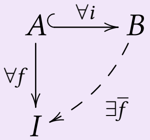

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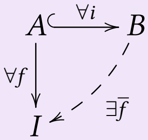

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Logical sheaves

The set $\mathbb{N}$ is **$\neg\neg$-separated** in that

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1. **Prop.** The group $(\mathbb{Q}/\mathbb{Z})^+$ has the Baer property.
2. **Lemma.** ZORN Let A be a group. Let $x_0 \in A$. Then there is an uncanonical map $i : A \to (\mathbb{Q}/\mathbb{Z})^+$ such that $i(x_0) = 0 \Rightarrow \neg\neg(x_0 = 0)$.
3. **Thm.** Every group canonically maps to a Baer group, and with ZORN, this map is injective.

Perspectives

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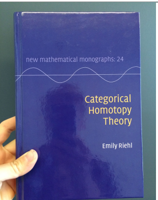

Plan:
Resolve issue by doing without injectives,
inspired by **Emily Riehl**.