– an invitation –

Connecting **inductive definitions**, **generic models** and a **modal multiverse** for algebra and combinatorics

REDCOM:
Reducing complexity in algebra, logic, combinatorics

Brixen
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Ingo Blechschmidt
University of Augsburg

Infinite data

Def. A set X together with a binary relation R is **almost-full$_\infty$** iff every infinite sequence $\alpha : \mathbb{N} \to X$ is **good** in that there exist numbers $i < j$ such that $\alpha(i)\ R\ \alpha(j)$.

Examples. $(\mathbb{N}, \leq)$, $X \times Y$ [Dickson], X^* [Higman], $\mathrm{Tree}(X)$ [Kruskal]

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Def. For a predicate P on finite lists over X, inductively define:

$$\frac{P(\sigma)}{P \mid \sigma} \qquad \frac{\forall x \in X.\ P \mid (\sigma :: x)}{P \mid \sigma}$$

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Def. A set X together with a binary relation R is **almost-full$_{\mathrm{ind}}$** iff $\mathrm{Good} \mid []$, where $\mathrm{Good}(\sigma) \equiv (\exists i < j.\ \sigma[i]\ R\ \sigma[j])$.

1. Constructively, almost-full$_{\mathrm{ind}}$ $\Rightarrow$ almost-full$_\infty$.
2. With LEM+DC, almost-full$_{\mathrm{ind}}$ $\Leftarrow$ almost-full$_\infty$.

Constructive red flags

A transitive relation $<$ on a set X is ...

- **well-founded$_\infty$** iff there is no infinite chain $x_0 > x_1 > \cdots$.
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- the theory of prime ideals of A proves "$x \in \mathfrak{p}$".

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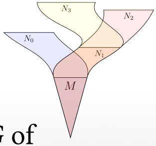
The set-theoretic multiverse

Def. A **model of set theory** is a (perhaps class-sized) structure $(M, \in)$ satisfying axioms such as those of ZFC.

Examples.



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- L, Gödel's constructible universe
- $V[G]$, a forcing extension containing a generic filter G of some poset of forcing conditions
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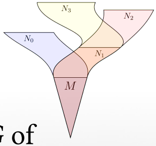

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Def. $\Diamond\varphi$ iff φ holds in **some extension** of the current universe.
$\Box\varphi$ iff φ holds in **all extensions** of the current universe.

- $\Box(\Diamond \mathrm{CH} \wedge \Diamond \neg\mathrm{CH})$, the continuum hypothesis is a **switch**
- $\Box\Diamond\Box(X \text{ is countable})$, existence of an enumeration is a **button**

Toposes and generic models

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3. Let $\mathbb{T}$ be a geometric theory. The **classifying topos** $\mathrm{Set}[\mathbb{T}]$ contains the **generic $\mathbb{T}$-model** $U_{\mathbb{T}}$. It is **conservative** in that for geometric implications φ, the following are equivalent:
   1. The statement φ holds for $U_{\mathbb{T}}$ in $\mathrm{Set}[\mathbb{T}]$.
   2. The statement φ holds for every $\mathbb{T}$-model in every topos.
   3. The statement φ is provable modulo $\mathbb{T}$.

$\mathbb{Z}$ $\qquad$ $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$ $\qquad$ $\mathcal{O}_X$ $\qquad$ $U_{\mathbb{T}}$

Modal algebra and combinatorics

Def. A statement φ holds …

1. **everywhere** ($\Box\varphi$) iff it holds in every (Grothendieck) topos (over the current base topos).
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   iff the *generic infinite descending chain* validates $\perp$
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NB: $(\Diamond\varphi) \Rightarrow \varphi$, if φ is a geometric implication.
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6. *Somewhere,* the law of excluded middle holds. [Barr]

Extracting constructive content

Prop. Let $(\leq)$ be a transitive almost-full$_{\text{ind}}$ relation. Then $(<)$, where $x < y \equiv (x \leq y \wedge \neg(y \leq x))$, is well-founded$_{\text{ind}}$.

Proof. Everywhere, there can be no infinite descending chain, as any such would also be good. □

Unrolling this proof gives a program $(\text{Good} \mid [\,]) \to \prod_{x:X} \text{Acc}(x)$.

```
data Acc {A : Set} (R : A → A → Set) : A → Set where
  acc : {x : A} → ((y : A) → R y x → Acc R y) → Acc R x

data _|_ {A : Set} (P : List A → Set) : List A → Set where
  here  : {σ : List A} → P σ → P | σ
  later : {σ : List A} → ((x : A) → P | (x ∷ σ)) → P | σ

module _ (A : Set) (_≼_ : A → A → Set) (≼-trans : Transitive _≼_) where
  Good : List A → Set
  Good σ = Σ[ i ∈ Fin (length σ) ] Σ[ j ∈ Fin (length σ) ]
      (i Data.Fin.< j) × (lookup σ j ≼ lookup σ i)

  _≴_ : A → A → Set
  x ≴ y = (x ≼ y) × ¬ (y ≼ x)

  lemma-no-good-descending-sequences : (σ : List A) → Good σ → Linked _≴_ σ → ⊥
  lemma-no-good-descending-sequences σ p q = { }0

  theorem : Good | [] → (x : A) → Acc _≴_ x
```

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3. Switching LEM on again, there is a minimal value $\beta(i(k_0))$ among all values of $\beta \circ i$. Hence γ is good in view of

$$\alpha(i(k_0)) \leq \alpha(i(k_0 + 1)), \qquad \beta(i(k_0)) \leq \beta(i(k_0 + 1)).$$

Outlook

I learned the idea to study a modal multiverse of toposes from **Alexander Oldenziel**, circa 2016. Foreshadowing results:

1984 André Joyal, Miles Tierney. *An extension of the Galois theory of Grothendieck.*

1987 Andreas Blass. *Well-ordering and induction in intuitionistic logic and topoi.*

2013 Shawn Henry. *Classifying topoi and preservation of higher order logic by geometric morphisms.*

Work by Milly Maietti and Steve Vickers on *arithmetic universes* is also closely connected. In progress:

- Develop details and formalize.
- Determine the precise list of valid modal principles.
- Carry out case studies with program extraction.
- Incorporate the right adjoints of geometric morphisms.