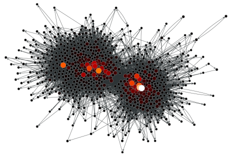
2023-11-28

Constructive forcing

These are the slides for a three-part lecture series for the autumn school *Proof and Computation* organized by Klaus Mainzer, Peter Schuster and Helmut Schwichtenberg, held in Herrsching from September 20th to September 16th, 2023.

We explore, in a constructive metatheory, the “Kripke–Joyal semantics of the internal language of Grothendieck toposes over sites given by preorders” together with applications in constructive algebra and combinatorics, but without presupposing familiarity with or using language from topos or category theory.



Let a continuous family of symmetric matrices be given:

$$\begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

Then for every parameter value $t \in \Omega$, classically there is

- a full list of eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ and
- an eigenvector basis $(v_1(t), \ldots, v_n(t))$.

Can locally the functions λ_i be chosen to be continuous?
How about the v_i?

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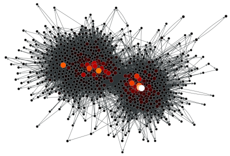

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Can locally the functions λ_i be chosen to be continuous? **Yes.**
How about the v_i? **No**

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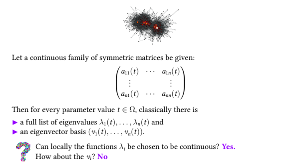
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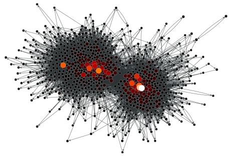

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- a full list of eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ and
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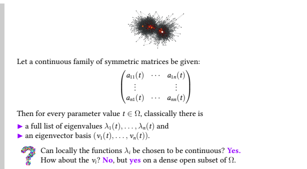
Can locally the functions λ_i be chosen to be continuous? **Yes.**
How about the v_i? **No**, but **yes** on a dense open subset of Ω.

2023-11-28

Constructive forcing



The slide presents a question on calculus. Constructive forcing establishes a connection with constructive linear algebra; we will understand that:

1. The eigenvalues (defined as the zeros of the characteristic polynomial) locally depend continuously on the parameter **because** it is a theorem of constructive linear algebra that symmetric matrices have a full list of eigenvalues.
2. The eigenvectors can **not** be expected to locally depend continuously **because** it is **not** a theorem of constructive linear algebra that symmetric matrices have an eigenvector basis.
3. There is a dense open subset of the parameter space which restores continuous dependence **because** it **is** a theorem of constructive linear algebra that every symmetric matrix does *not not* have an eigenvector basis.

A simple example where the eigenvectors cannot be chosen to locally depend continuously on the parameter is recorded here. Removing the origin there yields a suitable dense open subset.

A brief timeline

1878 · · · · · Cantor advances the **continuum hypothesis**, the claim that $2^{\aleph_0} = \aleph_1$.

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Gödel's proof is by the L-**translation**, where L is the "constructible universe". This translation "relativizes quantification to L", for instance the L-translation of

$$\varphi :\equiv \big(\forall x.\, \exists y.\, \ldots\big) \qquad \text{is} \qquad \varphi^L \equiv \big(\forall (x \in L).\, \exists (y \in L).\, (\ldots)^L\big).$$

We can then verify, in a weak metatheory such as PRA, that for every formula φ in the language of set theory: If ZFC+CH $\vdash \varphi$, then ZF $\vdash \varphi^L$.

Specializing to $\varphi :\equiv \bot$ we obtain in particular: If ZFC is inconsistent, then so is ZF. The axiom of choice does not introduce new inconsistencies.

In modern semantic language: While the axiom of choice and CH might fail in the base universe V (= the class of all sets), they always hold in L. Gödel's L was the first *inner model* (= class-sized model of set theory) explicitly studied, nowadays we know many.

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For his proof, Cohen invented the technique of *forcing*, situated in classical mathematics where the base universe V is assumed to validate the axioms of ZFC.

Recall that a given ring R or group can be extended in various ways, to include "generic elements" as in $R[X]$ or elements with prescribed relations as in $R[X]/(X^2+1) =: R[i]$. The idea of forcing is to construct similar such extensions, but not of rings but of universes (traditionally set-sized models of ZF or ZFC, but also class-sized models, or models of intuitionistic set theories, or models of type theories, or even models of arithmetic).

In semantic language, from a high level the idea of Cohen's independency proof is the following: Whether the base universe V contains a cardinal number intermediate between $\aleph_0$ and $2^{\aleph_0}$ is uncertain. But there is a certain extension of the base universe—constructed by forcing—which does contain such a number. Like the base V, this forcing extension still validates the axioms of ZFC. Hence there cannot be a ZFC-proof of CH, as in Cohen's extension ¬CH holds.

Syntactically, Cohen's forcing provides us with an explicit formula translation $\varphi \mapsto \varphi^C$ such that PRA proves: For every formula φ, if $\text{ZFC}+\neg\text{CH} \vdash \varphi$, then $\text{ZFC} \vdash \varphi^C$.

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Joel David Hamkins argues: In view of our rich experience with worlds which validate CH and worlds which don't, we shouldn't be surprised that no proposed new axiom for settling CH is ultimately convincing.

Instead, we should embrace the multiverse of all models of set theory and explore how the truth values of statements of interest change when we travel the multiverse (for instance, by passing from a universe to one of its forcing extensions).

In this generalized sense, the continuum hypothesis is settled: We have a good understanding of the stability properties of CH under important constructions. In particular, for a certain precise meaning of "universe" and "extension", we know that CH is a *switch*: $\Box(\Diamond \text{CH} \wedge \Diamond \neg\text{CH})$; in words: Every universe can be extended both to a universe in which CH holds and to a universe in which ¬CH holds.

An exposition and references for further reading about the multiverse position can be found here.

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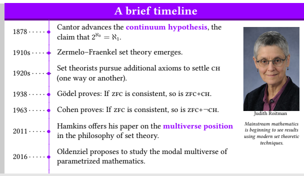
Judith Roitman

Mainstream mathematics is beginning to see results using modern set theoretic techniques.



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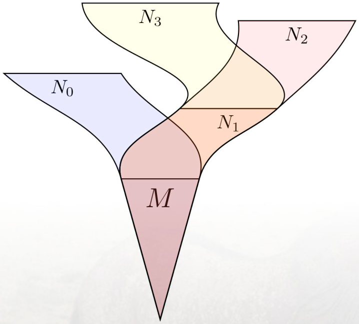
J. Roitman, The uses of set theory, *Math. Intelligencer* **14**(1) (1992), 63–69.

Forcing is useful not only to explore the range of foundational possibility; it has many more applications across several subjects of mathematics.

In particular, we will discuss applications of the constructive version of classical set-theoretic forcing in constructive algebra and combinatorics.

Constructive forcing



Extending the universe in various ways,

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Constructive forcing

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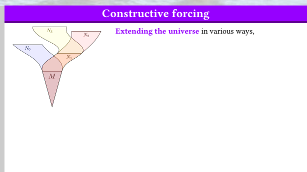

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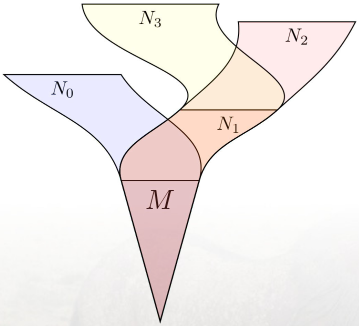

Extending the universe in various ways,

similarly how we can extend groups or rings,

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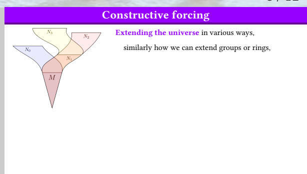

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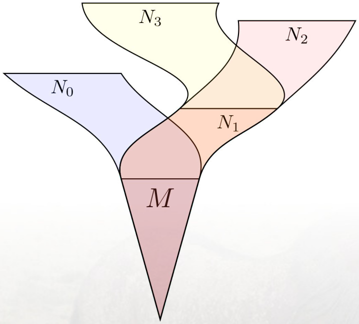

Extending the universe in various ways,

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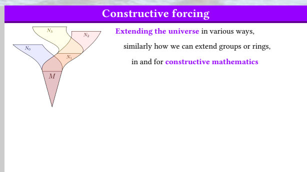

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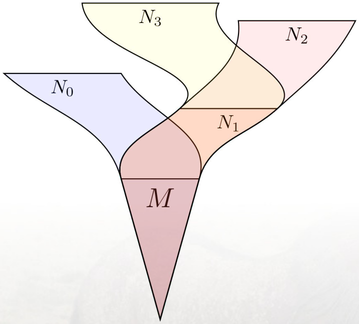

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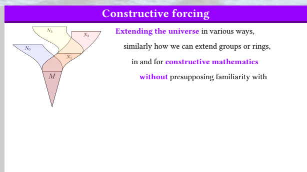

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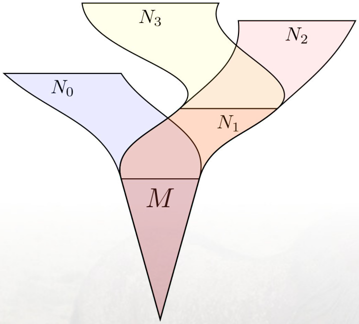

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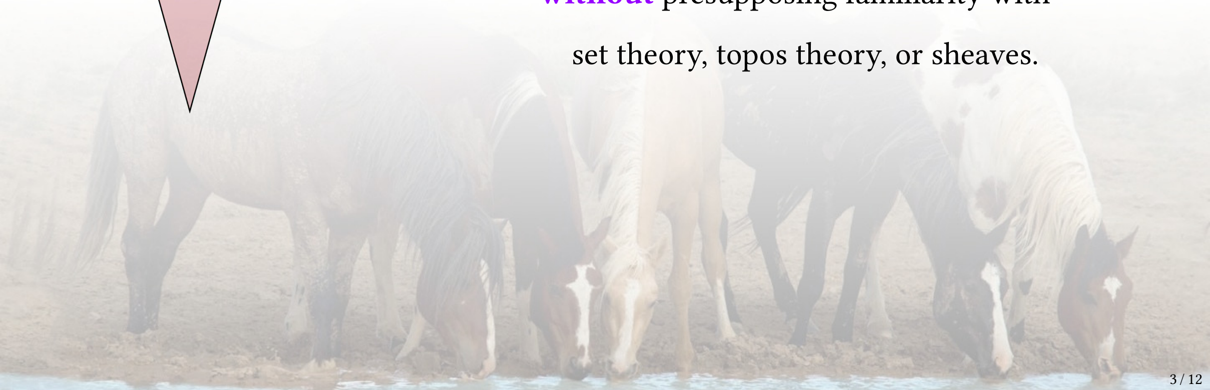

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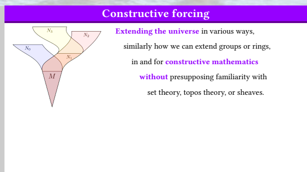

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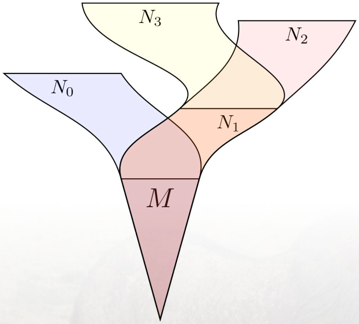

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Outline:

1. What can forcing do for you?
2. Forcing notions and Kripke–Joyal semantics
3. Case studies in constructive algebra and combinatorics

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Constructive forcing

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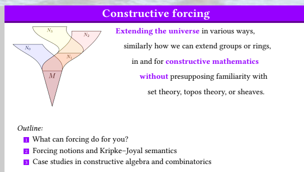
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What can forcing do for you?

1. Explore foundational possibility

There are forcing extensions with
CH, ¬CH, LEM, ¬LEM, …

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Forcing ¬CH was the historical use case for forcing.

With constructive forcing, where we do not blanketly assume that the base universe validates LEM, we can inquire the status of LEM; a result in this direction is that LEM is, like CH in the context of ZFC, a *switch*:

The base universe can always be extended in such a way as to force ¬LEM (in fact, most forcing extensions will falsify LEM even in case the base universe validates it), and also in such a way to force LEM. The latter provides us with a semantic view of one of the techniques for extracting constructive content from classical proofs, namely the double-negation translation combined by the continuation trick.

What can forcing do for you?

1. Explore foundational possibility

There are forcing extensions with CH, ¬CH, LEM, ¬LEM, …

2. Demonstrate unprovability

The fundamental theorem of algebra is **not constructively provable** as there is a forcing extension where **it is false**.

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We will discuss the particular example of the (naive formulation of the) fundamental theorem of algebra below.

Forcing has been used to construct countermodels to various questions of constructive (reverse) mathematics, too many to list here. To give just one pointer, a countermodel for the classical implication "if there is no infinite descending chain, then the partial order is inductively well-founded" is presented here: A. Blass, Well-ordering and induction in intuitionistic logic and topoi, in: *Mathematical Logic and Theoretical Computer Science*. Ed. by D. Kueker, E. Lopez-Escobar, and C. Smith. Vol. 106. Lect. Notes Pure Appl. Math. Marcel Dekker, 1987, pp. 29–48.

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The fundamental theorem of algebra is **not constructively provable** as there is a forcing extension where **it is false**.

3. Harness convenient fictions

For every set, there is a forcing extension where it is **countable**.

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The real numbers don't contain a number i such that $i^2 = 0$. Still, for many results in real analysis, it is convenient to broaden our notion of existence and pass to the complex plane; the imaginary unit is a *mathematical phantom* in the sense of Gavin Wraith, a useful tool helping us deduce results about real numbers. Nowadays there are few ontological concerns about the imaginary unit: We understand that, in the end, every statement about complex numbers can be recast as a statement about pairs of real numbers.

In exactly the same fashion, the objects furnished by forcing can be understood as useful fictions. We will discuss how statements about the forcing extension can be recast as statements about the base universe.

Given an inhabited set X in the base universe, most pronouncedly a set which is uncountable or for which no surjection $\mathbb{N} \twoheadrightarrow X$ can be efficiently evaluated, a particularly tantalizing fiction is the *generic surjection* $\mathbb{N} \twoheadrightarrow X$. It exists in a custom-tailored forcing extension of the base universe and is useful to apply tools made for the countable setting to the uncountable; we will discuss an example on the next slide.

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A preorder X is well iff the **generic sequence** $\mathbb{N} \to X$ is good.

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A preorder X is well iff the generic sequence $\mathbb{N} \to X$ is good.

Typically there are already plenty of maps $\mathbb{N} \to X$ in the base universe; hence constructing a forcing extension which contains a “fresh” such map—the so-called *generic sequence*—is not something which would usually be contemplated in classical set-theoretic forcing.

In the context of constructive mathematics, however, the generic sequence turns out to be quite useful. It can be used to cast in a familiar naive language—the language of infinite sequences—definitions, results and proofs from constructive combinatorics which use constructively more appropriate inductively defined notions. We will discuss an example below.

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5. Study parametric mathematics

Eigenvectors depend continuously on the parameter iff, in a suitable forcing extension, they merely exist.

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Slides by Matthias Hutzler: Introduction to synthetic algebraic geometry

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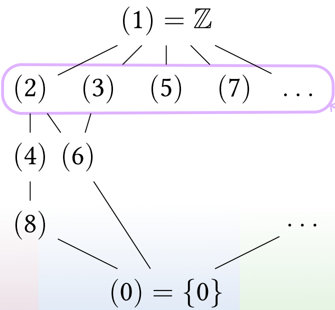

maximal among the proper ideals

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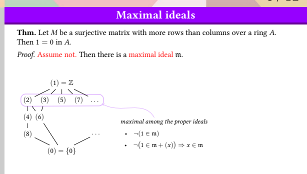

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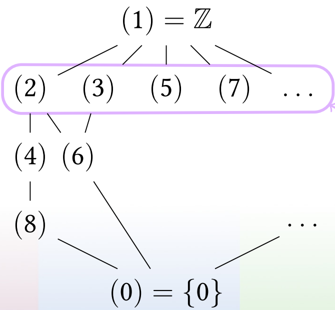

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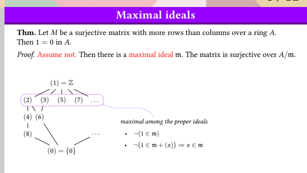

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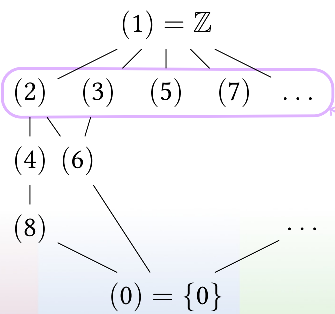

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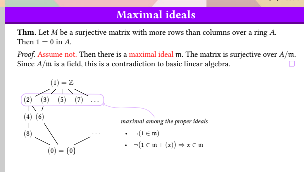

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The slide presents a standard proof as offered by most textbooks on commutative algebra. The proof is quite efficient from a viewpoint of mathematical organization, as it quickly succeeds in reducing to the field situation. As such, it is short and memorable.

However, the proof can also be critized for appealing to the transfinite two times; the methods of the proof are at odds with the concreteness of the statement of the theorem—from given equations witnessing surjectivity, $Mv_i = e_i$, we are asked to deduce the equation $1 = 0$.

For this reason, the theorem and its classical proof are often used as case studies for tools and techniques aiming to extract constructive content from classical proofs. One such technique employs constructive forcing. The first(?) constructive proof, found directly without using extraction techniques, is laid out in a beautiful short note by Richman.

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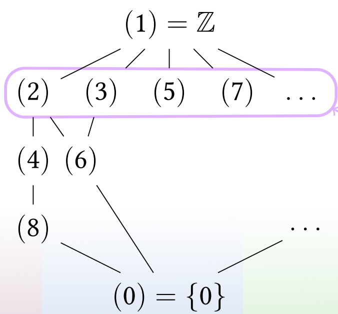

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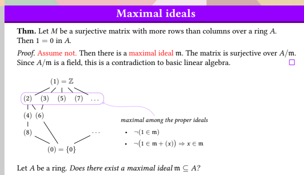

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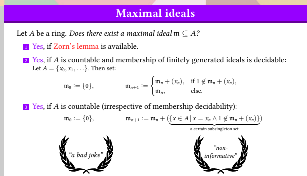

The iterative construction in the countable case without decidability assumptions is due to Krivine; it was later clarified by Berardi and Valentini. It is a parlor trick, resulting in a subset which formally verifies the axioms for a maximal ideal but without carrying out any actual work. Indeed, the resulting ideal will in general not be a detachable subset of the ring.

Surprisingly, there is still computational content in this construction, as explored in this joint paper with Peter Schuster; one interpretation of our observation is that classical proofs don't “really” require a maximal ideal; they just use that notion for structuring hidden computations.

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Unwinding all the definitions from constructive forcing and from the iterative maximal ideal construction, and eliminating the application of LEM from the classical proof presented before, we mechanically arrive at the constructive direct proof presented on the slide.

Infinite data

7, 4, 3, ...

2023-11-28

Constructive forcing
└─What can forcing do for you?
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The presented proof rests on the law of excluded middle and hence cannot immediately be interpreted as a program for finding suitable indices $i < j$. However, constructive proofs are also possible (for instance by induction on the value of a given term of the sequence), and furthermore constructive proofs can be extracted from the presented classical proof.

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The class of well preorders is stable under cartesian products, lists and trees, by Dickson's Lemma, Higson's Lemma and Kruskal's Theorem, respectively. However, in their naive formulations, these are merely theorems of classical mathematics. For general constructive results, the definition of "well" needs to be improved.

Classical texts often employ needless negations in their definitions; this is an XXX

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Infinite data

$$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$$

$\leq$

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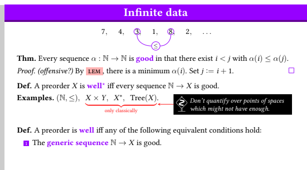

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Ingredients for forcing

To construct a forcing extension, we require:

1. a base universe V
2. a preorder L of **forcing conditions** in V, pictured as **finite approximations** (*convention:* $\tau \preccurlyeq \sigma$ means that τ is a better finite approximation than σ)
3. a **covering system** governing how finite approximations evolve to better ones (for each $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$, with a simulation condition)

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For the generic prime ideal of a ring A

Use **f.g. ideals** as forcing conditions, where $\mathfrak{b} \preccurlyeq \mathfrak{a}$ iff $\mathfrak{b} \supseteq \mathfrak{a}$, and be prepared to grow $\mathfrak{a}$ to …

(a) one of $\emptyset$, if $1 \in \mathfrak{a}$, to make $\mathfrak{a}$ more proper

(b) one of $\{\mathfrak{a} + (x), \mathfrak{a} + (y)\}$, if $xy \in \mathfrak{a}$, to make $\mathfrak{a}$ more prime

The eventually monad

Let L be a forcing notion.

Let P be a monotone predicate on L (if $\tau \preccurlyeq \sigma$, then $P\sigma \Rightarrow P\tau$).
For instance, in the case $L = X^*$:

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We then define "$P \mid \sigma$" ("P bars σ") inductively by the following clauses:

1. If $P\sigma$, then $P \mid \sigma$.
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The ∇-translation

For bounded first-order formulas over the (large) first-order signature which has

1. one sort $\underline{X}$ for each set X in the base universe,
2. one n-ary function symbol $\underline{f} : \underline{X_1} \times \cdots \times \underline{X_n} \to \underline{Y}$ for each map $f : X_1 \times \cdots \times X_n \to Y$,
3. one n-ary relation symbol $\underline{R} \hookrightarrow \underline{X_1} \times \cdots \times \underline{X_n}$ for each relation $R \subseteq X_1 \times \cdots \times X_n$, and
4. an additional unary relation symbol $G \hookrightarrow \underline{L}$ (for the *generic filter* of L),

we recursively define:

$\sigma \models s = t$	iff	$\nabla\sigma.\, [\![s]\!] = [\![t]\!]$.	$\sigma \models \underline{R}(s_1, \ldots, s_n)$	iff	$\nabla\sigma.\, R([\![s_1]\!], \ldots, [\![s_n]\!])$.
$\sigma \models \varphi \Rightarrow \psi$	iff	$\forall(\tau \preccurlyeq \sigma).\, (\tau \models \varphi) \Rightarrow (\tau \models \psi)$.	$\sigma \models G\tau$	iff	$\nabla\sigma.\, \sigma \preccurlyeq [\![\tau]\!]$.
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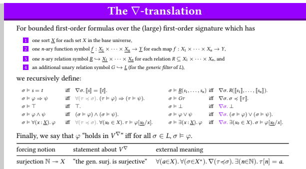
$\sigma \models \forall(x : \underline{X}).\, \varphi$	iff	$\forall(\tau \preccurlyeq \sigma).\, \forall(x_0 \in X).\, \tau \models \varphi[\underline{x_0}/x]$.	$\sigma \models \exists(x : \underline{X}).\, \varphi$	iff	$\nabla\sigma.\, \exists(x_0 \in X).\, \sigma \models \varphi[\underline{x_0}/x]$.

Finally, we say that φ "holds in V^∇" iff for all $\sigma \in L$, $\sigma \models \varphi$.

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(a \in X).\, \forall(\sigma \in X^*).\, \nabla(\tau \preccurlyeq \sigma).\, \exists(n \in \mathbb{N}).\, \tau[n] = a$.



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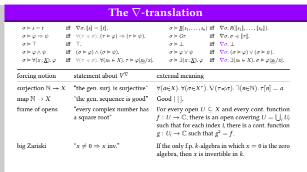
map $\mathbb{N} \to X$	"the gen. sequence is good"	Good $\mid [\,]$.
frame of opens	"every complex number has a square root"	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.



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\end{array}
$$

forcing notion	statement about V^∇	external meaning
surjection $\mathbb{N} \twoheadrightarrow X$	"the gen. surj. is surjective"	$\forall(a{\in}X).\, \forall(\sigma{\in}X^*).\, \nabla(\tau{\preccurlyeq}\sigma).\, \exists(n{\in}\mathbb{N}).\, \tau[n] = a.$
map $\mathbb{N} \to X$	"the gen. sequence is good"	Good $\mid [\,]$.
frame of opens	"every complex number has a square root"	For every open $U \subseteq X$ and every cont. function $f : U \to \mathbb{C}$, there is an open covering $U = \bigcup_i U_i$ such that for each index i, there is a cont. function $g : U_i \to \mathbb{C}$ such that $g^2 = f$.
big Zariski	"$x \neq 0 \Rightarrow x$ inv."	If the only f.p. k-algebra in which $x = 0$ is the zero algebra, then x is invertible in k.
little Zariski	"every f.g. vector space does *not not* have a basis"	**Grothendieck's generic freeness lemma**

Outlook

Passing to and from extensions

Thm. Let φ be a **bounded first-order formula** not mentioning G. In each of the following situations, we have that φ holds in V^∇ iff φ holds in V:

1. L and all coverings are inhabited (proximality).
2. L contains a top element, every covering of the top element is inhabited, and φ is a coherent implication (positivity).

The mystery of nongeometric sequents

The **generic ideal** of a ring is maximal:

$$(x \in \mathfrak{a} \Rightarrow 1 \in \mathfrak{a}) \Longrightarrow 1 \in \mathfrak{a} + (x).$$

The **generic ring** is a field:

$$(x = 0 \Rightarrow 1 = 0) \Longrightarrow (\exists y.\, xy = 1).$$

Traveling the multiverse …

LEM is a **switch** and **holds positively**; being countable is a **button**.

Every instance of DC **holds proximally**.

A geometric implication is provable iff it holds **everywhere**.

… upwards, but always keeping ties to the base.

2023-11-28

Constructive forcing
└─Basics of forcing
└─Outlook

Outlook
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Formalities

Def. A **forcing notion** consists of a preorder L of **forcing conditions**, and for every $\sigma \in L$, a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$ of **coverings** of σ such that: If $\tau \preccurlyeq \sigma$ and $R \in \mathrm{Cov}(\sigma)$, there should be a covering $S \in \mathrm{Cov}(\tau)$ such that $S \subseteq \downarrow R$.

	preorder L	coverings of an element $\sigma \in L$	filters of L
1	X^*	$\{\sigma x \mid x \in X\}$	maps $\mathbb{N} \to X$
2	X^*	$\{\sigma x \mid x \in X\}$, $\{\sigma\tau \mid \tau \in X^*, a \in \sigma\tau\}$ for each $a \in X$	surjections $\mathbb{N} \twoheadrightarrow X$
3	f.g. ideals	—	ideals
4	f.g. ideals	$\{\sigma + (a), \sigma + (b)\}$ for each $ab \in \sigma$, $\{\}$ if $1 \in \sigma$	prime ideals
5	opens	$\mathcal{U}$ such that $\sigma = \bigcup \mathcal{U}$	points
6	$\{\star\}$	$\{\star \mid \varphi\} \cup \{\star \mid \neg\varphi\}$	witnesses of LEM

Def. A *filter* of a forcing notion (L, Cov) is a subset $F \subseteq L$ such that

1. F is upward-closed: if $\tau \preccurlyeq \sigma$ and if $\tau \in F$, then $\sigma \in F$;
2. F is downward-directed: F is inhabited, and if $\alpha, \beta \in F$, then there is a common refinement $\sigma \preccurlyeq \alpha, \beta$ such that $\sigma \in F$; and
3. F splits the covering system: if $\sigma \in F$ and $R \in \mathrm{Cov}(\sigma)$, then $\tau \in F$ for some $\tau \in R$.

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