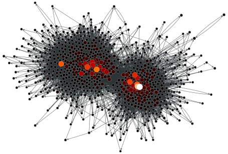

Let a continuous family of symmetric matrices be given:

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There for every parameter value t, classically there is

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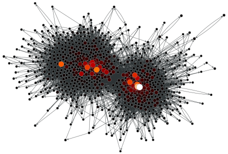

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How about the v_i? **No.**

– an invitation –

New modal operators for constructive mathematics

Type Theory, Constructive Mathematics and Geometric Logic

CIRM
May 2nd, 2023

Ingo Blechschmidt
j.w.w. Alexander Oldenziel

Questions

1. Why has the inductive revolution been so powerful?
2. Why do proofs using Zorn's maximal ideals work so well in constructive algebra?
3. Why are elements of $\bigcap_{\mathfrak{p}} \mathfrak{p}$ not necessarily nilpotent?
4. How can we extract computational content from classical proofs?

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$$7, \quad 4, \quad 3, \quad \ldots$$

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Examples. $(\mathbb{N}, \leq)$, $\underbrace{X \times Y, \quad X^*, \quad \mathrm{Tree}(X)}$.

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Can we extract a constructive proof that well preorders are well-founded?

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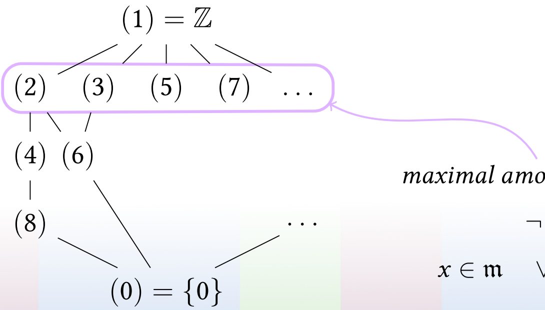

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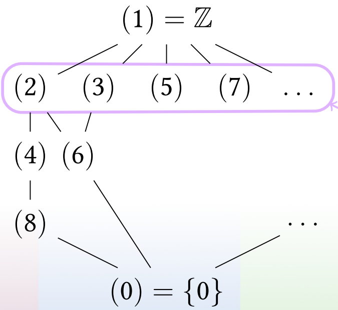

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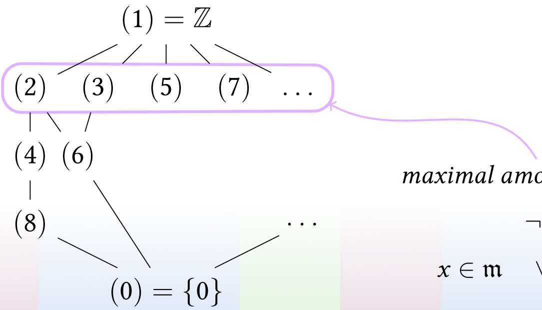

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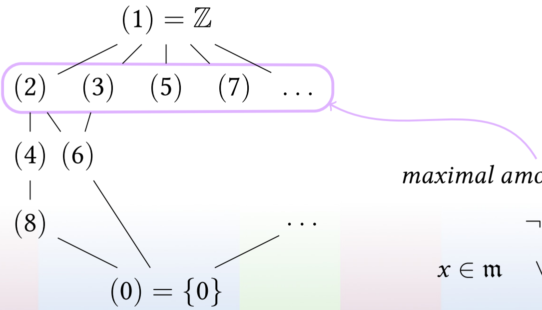

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"a bad joke"

"non-informative"

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4. In the general case: No,
but *first-order consequences* of the existence of a maximal ideal do hold.

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Constructive forcing (= Grothendieck toposes)

Let L be a **forcing notion**, a preorder equipped with a **covering system**.[1] **Filters** $F \subseteq L$ are subsets which are upward-closed, downward-directed and split the covering system.[2]

[1] A covering system consists of a set $\mathrm{Cov}(\sigma) \subseteq P(\downarrow\sigma)$ of *coverings* for each element $\sigma \in L$ subject only to the following simulation condition: If $\tau \preccurlyeq \sigma$ and $R \in \mathrm{Cov}(\sigma)$, there should be a covering $S \in \mathrm{Cov}(\tau)$ such that $S \subseteq \downarrow R$.

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3	f.g. ideals	—	ideals
4	f.g. ideals	$\{\sigma + (a), \sigma + (b)\}$ for each $ab \in \sigma$, $\{\}$ if $1 \in \sigma$	prime ideals
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Def. Given a monotone predicate P on L, inductively define:

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We use quantifier-like notation: "$\nabla\sigma.\, P\sigma$" means $P \mid \sigma$.

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4. Let X be a preorder. Then:
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   - $(<)$ is well-founded iff the generic strictly descending chain validates $\bot$.

The modal multiverse

In general, "φ holds in V^∇" and "φ holds in V" are *not* equivalent.

- For **positive** extensions, they are equivalent for coherent implications.
  – e.g. the "Barr cover".
- For **positive overt** extensions, they are equivalent for bounded first-order formulas.
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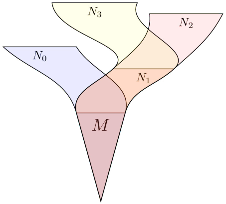

Foreshadowed by:

1984 André Joyal, Miles Tierney. *An extension of the Galois theory of Grothendieck.*
1987 Andreas Blass. *Well-ordering and induction in intuitionistic logic and topoi.*
2010s Milly Maietti, Steve Vickers. Ongoing work on arithmetic universes.
2011 Joel David Hamkins. *The set-theoretic multiverse.*
2013 Shawn Henry. *Classifying topoi and preservation of higher order logic by geometric morphisms.*

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For every inhabited set X, *proximally* there is an enumeration $\mathbb{N} \twoheadrightarrow X$.

For every ring, *proximally* there is a maximal ideal.

A preorder is well iff *everywhere*, every sequence is good.

A relation is well-founded iff *everywhere*, there is no descending chain.

A ring element is nilpotent iff all prime ideals *everywhere* contain it.

Somewhere, the law of excluded middle holds.

The multiverse perspective

1. *Why has the inductive revolution been so powerful?*
   Because the inductive conditions are equivalent to truth in *all* forcing extensions.

2. *Why do proofs using Zorn's maximal ideals work so well in constructive algebra?*
   Because every ring proximally has a maximal ideal.

3. *Why are elements of $\bigcap_{\mathfrak{p}} \mathfrak{p}$ not necessarily nilpotent?*
   Because we forgot the prime ideals in forcing extensions.

4. *How can we extract computational content from classical proofs?*
   By traveling the multiverse (upwards, keeping ties to the base), exploiting that
   – LEM holds *somewhere* and
   – DC holds *proximally*.

```
module _ (A : Set) where

open import Data.List
open import Data.List.Membership.Propositional
open import Data.Product

data Eventually (P : List A → Set) : List A → Set where
  now
    : {σ : List A}
    → P σ
    → Eventually P σ
  later
    : {σ : List A} {a : A}
    → ((τ : List A) → a ∈ (σ ++ τ) → Eventually P (σ ++ τ))
    → Eventually P σ
```

```
U:**-  Countable.agda   All L1    <N>    (Agda:Checked +5 Undo-Tree)

U:%*-  *All Done*   All L1    <M>    (AgdaInfo Undo-Tree)
```

Partial Agda formalization available.