The secret of the number 5

Ingo Blechschmidt

36th Chaos Communication Congress

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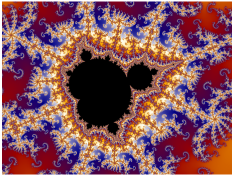
Dedicated to Prof. Dr. Jost-Hinrich Eschenburg.



Outline

A design pattern in nature

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Fibonacci numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

The number of spirals on a sunflower is always a Fibonacci number (or a number very close to a Fibonacci number), for instance in the large picture on the previous slide there are 21 clockwise spirals and 34 counterclockwise ones. Why?

A curious fraction

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Multiplying by the denominator, we obtain $1 = 2x + x^2$, so we only have to solve the quadratic equation $0 = x^2 + 2x - 1$, thus

$$x = \frac{-2 + \sqrt{8}}{2} = -1 + \sqrt{2} \quad \text{or} \quad x = \frac{-2 - \sqrt{8}}{2} = -1 - \sqrt{2}.$$

It's the positive possibility.

More examples

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = \sqrt{2}$$

$$2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ddots}}} = \sqrt{5}$$

$$3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \ddots}}} = \sqrt{10}$$

More examples

$$[1; 2, 2, 2, \ldots] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = \sqrt{2}$$

$$[2; 4, 4, 4, \ldots] = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ddots}}} = \sqrt{5}$$

$$[3; 6, 6, 6, \ldots] = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \ddots}}} = \sqrt{10}$$

More examples

1. $\sqrt{2} = [1; 2, 2, 2, 2, 2, 2, 2, 2, \ldots]$
2. $\sqrt{5} = [2; 4, 4, 4, 4, 4, 4, 4, 4, \ldots]$
3. $\sqrt{10} = [3; 6, 6, 6, 6, 6, 6, 6, 6, \ldots]$
4. $\sqrt{6} = [2; 2, 4, 2, 4, 2, 4, 2, 4, \ldots]$
5. $\sqrt{14} = [3; 1, 2, 1, 6, 1, 2, 1, 6, \ldots]$
6. $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$

The digits of the number $e = 2.7182818284\ldots$, the basis of the natural logarithm, do not have any concernible pattern. But its continued fraction expansion is completely regular.

The Euclidean algorithm

Recall $\sqrt{2} = [1; 2, 2, 2, \ldots] = 1.41421356\ldots$

$$\begin{aligned}
1.41421356\ldots &= 1 \cdot 1.00000000\ldots + 0.41421356\ldots \\
1.00000000\ldots &= 2 \cdot 0.41421356\ldots + 0.17157287\ldots \\
0.41421356\ldots &= 2 \cdot 0.17157287\ldots + 0.07106781\ldots \\
0.17157287\ldots &= 2 \cdot 0.07106781\ldots + 0.02943725\ldots \\
0.07106781\ldots &= 2 \cdot 0.02943725\ldots + 0.01219330\ldots \\
0.02943725\ldots &= 2 \cdot 0.01219330\ldots + 0.00505063\ldots \\
&\vdots
\end{aligned}$$

Why does the Euclidean algorithm give the continued fraction coefficients? Let's write

$$\begin{aligned} x &= a_0 \cdot 1 \ + r_0 \\ 1 &= a_1 \cdot r_0 + r_1 \\ r_0 &= a_2 \cdot r_1 + r_2 \\ r_1 &= a_3 \cdot r_2 + r_3 \end{aligned}$$

and so on, where the numbers a_n are natural numbers and the residues r_n are smaller than the second factor of the respective adjacent product. Then:

$$\begin{aligned} x &= a_0 + r_0 = a_0 + 1/(1/r_0) \\ &= a_0 + 1/(a_1 + r_1/r_0) = a_0 + 1/(a_1 + 1/(r_0/r_1)) \\ &= a_0 + 1/(a_1 + 1/(a_2 + r_2/r_1)) = \cdots \end{aligned}$$

In the beautiful language Haskell, the code for lazily calculating the infinite continued fraction expansion is only one line long (the type declaration is optional).

```
cf ::  Double -> [Integer]
cf x = a :  cf (1 / (x - fromIntegral a)) where a = floor x
```

So the continued fraction expansion of a number x begins with a, the integral part of x, and continues with the continued fraction expansion of $1/(x-a)$.

Note that because of floating-point inaccuracies, only the first few terms of the expansion are reliable. For instance, `cf (sqrt 6)` could yield

$$[2,2,4,2,4,2,4,2,4,2,4,2,4,2,4,2,2,1,48,2,4,6,1,\ldots].$$

Best approximations using continued fractions

Theorem

Cutting off the infinite fraction expansion of a number x yields a fraction a/b which is closest to x under all fractions with denominator $\leq b$.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} \rightsquigarrow 1 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}} = \frac{17}{12} \approx 1.42$$

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Bonus. The bigger the coefficient after the cut-off is, the better is the approximation a/b.

More precisely, the bonus statement is that the distance from x to a/b is less than $1/(a_n a_{n+1})$, where a_n is the last coefficient to be included in the cut-off and a_{n+1} is the first coefficient after the cut-off.

Love is important.

♡

Pi is important.

π

Approximations of π

$$\pi = 3.1415926535\ldots = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ddots}}}}$$

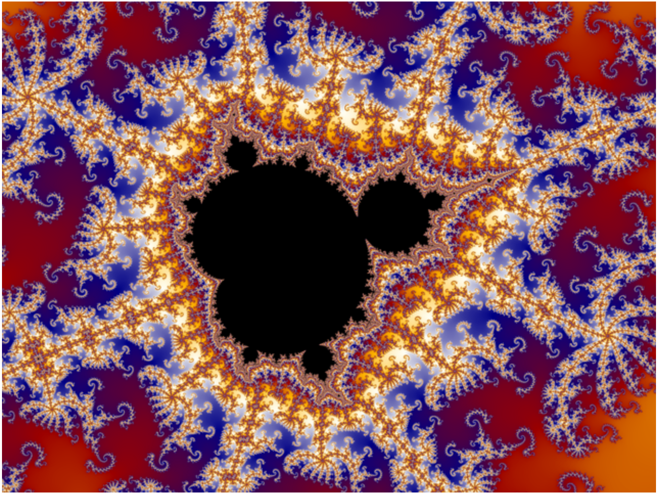
1. 3
2. $[3; 7] = 22/7 = \underline{3.14}28571428\ldots$
3. $[3; 7, 15] = 333/106 = \underline{3.1415}094339\ldots$
4. $[3; 7, 15, 1] = 355/113 = \underline{3.141592}9203\ldots$ (Milü)

We do not know for sure how people in ancient times calculated approximations to π. But one possibility is that they used some form of the Euclidean algorithm (of course not using decimal expansions, but for instance strings of various lengths).

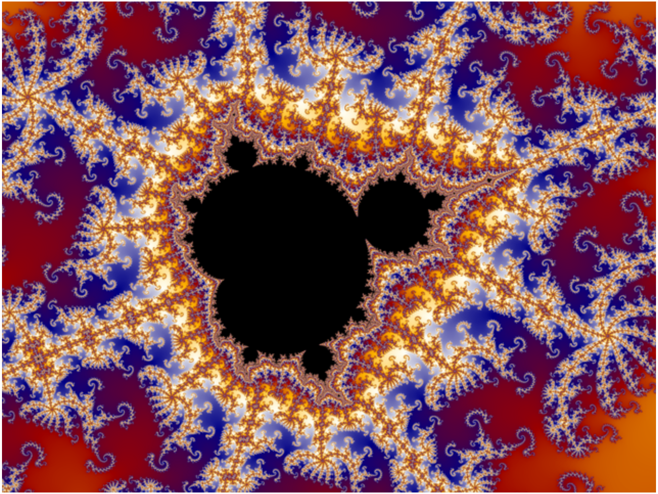
Because the coefficient 292 appearing in the continued fraction expansion of π is exceptionally large, the approximation $355/113$ is exceptionally good. That's a nice mathematical accident! I like to think that better approximations were not physically obtainable in ancient times, but thanks to this accident the best approximation that was obtainable was in fact an extremely good one. In particular, it's much better than the denominator 113 might want us to think.

NB: The fraction $355/113$ is easily memorized (11–33–55).

The Mandelbrot fractal



The Mandelbrot fractal



The Fibonacci numbers show up in the Mandelbrot fractal.

See `http://math.bu.edu/DYSYS/FRACGEOM2/node7.html` for an explanation of where and why the Fibonacci numbers show up in the Mandelbrot fractal.

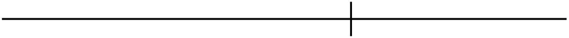
Spirals in nature

The most irrational number

For plants, the optimal angle of consecutive seeds is not …

- $90^\circ = \frac{1}{4} \cdot 360^\circ$ nor is it
- $45^\circ = \frac{1}{8} \cdot 360^\circ$.

Rather, it is the **golden angle** $\Phi \cdot 360^\circ \approx 582^\circ$ (equivalently 222°), where Φ is the **golden ratio**:
$\Phi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$



Theorem

The golden ratio Φ is the ***most irrational number****.*

Proof. $$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}.$$

The golden ratio appears in lots of places in nature and art. If you divide a segment in the golden ratio, the longer subsegment will be Φ times as long as the shorter subsegment; more conceptually:

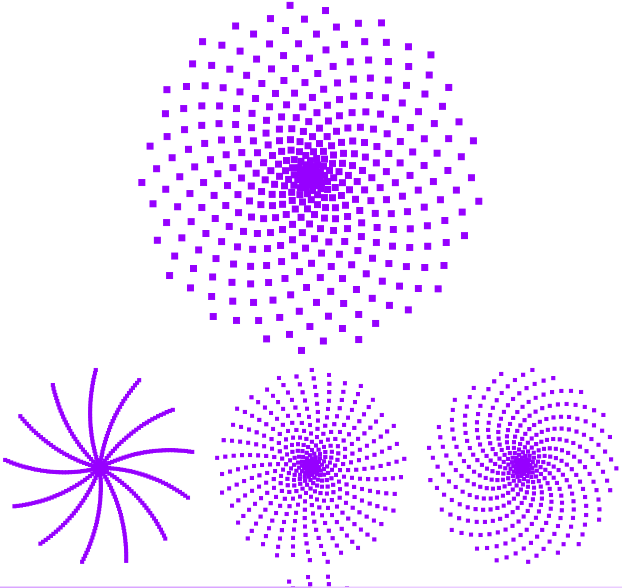
$$\text{total segment} : \text{longer subsegment} = \text{longer subsegment} : \text{shorter subsegment}.$$

If you use a fraction $\frac{a}{b}$ of the full circle as rotation angle, then after b turns you'll arrive at exactly the same location as you started. That's bad! Space is wasted this way.

It's better to use a number which can *not* be expressed as a fraction – an *irrational number.* Of all irrational numbers, one should pick the *most irrational* one.

Recall that a number can the better be approximated by fractions the larger the coefficients in the continued fraction expansion are. With Φ, the coefficients are as small as possible. This is the reason why Φ is the "most irrational" number. It is the hardest number to approximate by fractions.

(Not) using the golden angle

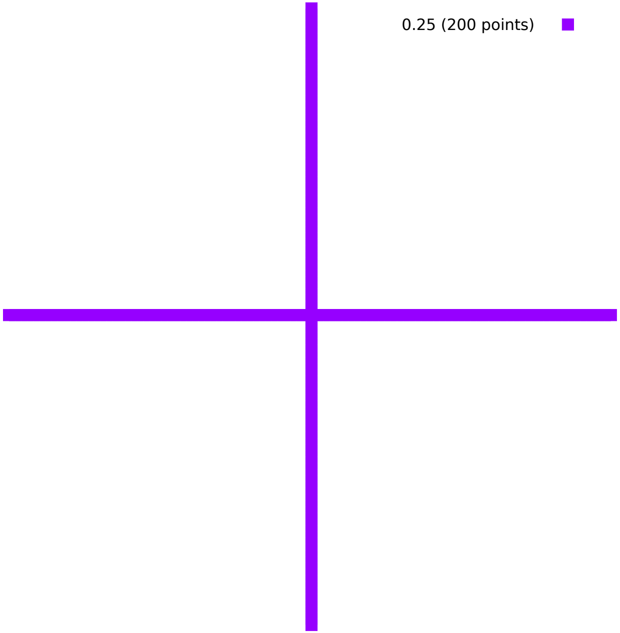


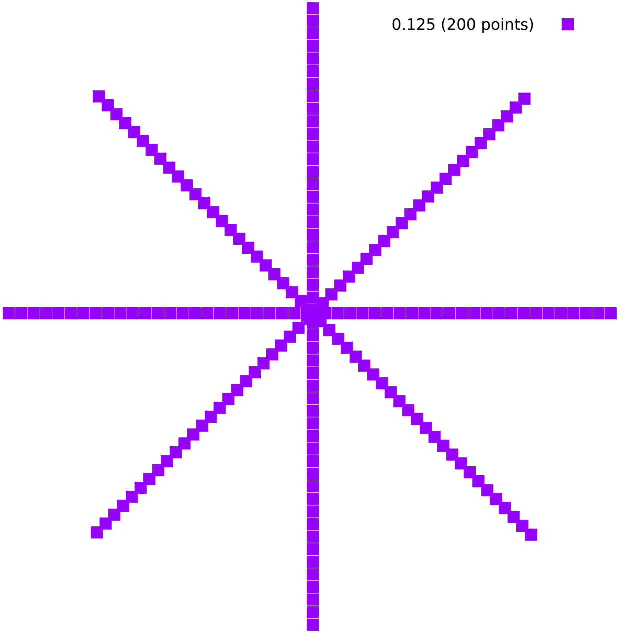
The top figure uses the golden angle. The angles used in the four figures in the bottom are:

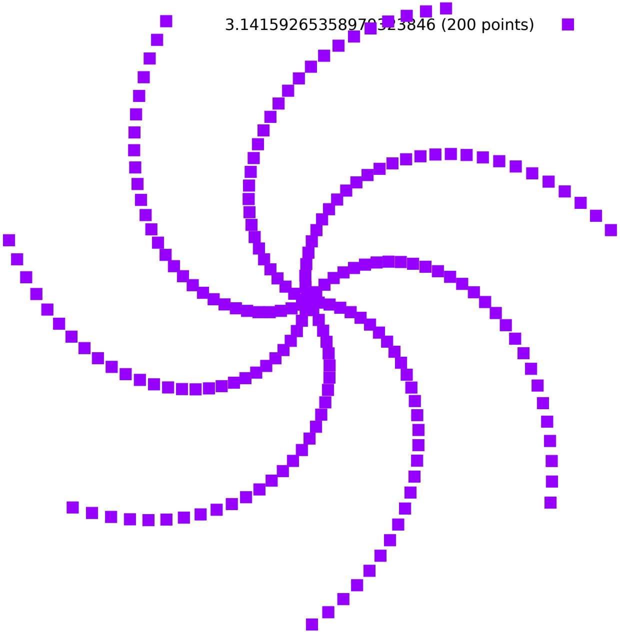
1. golden angle $- 1^\circ$
2. golden angle $- 0.1^\circ$
3. golden angle $+ 0.1^\circ$
4. golden angle $+ 1^\circ$

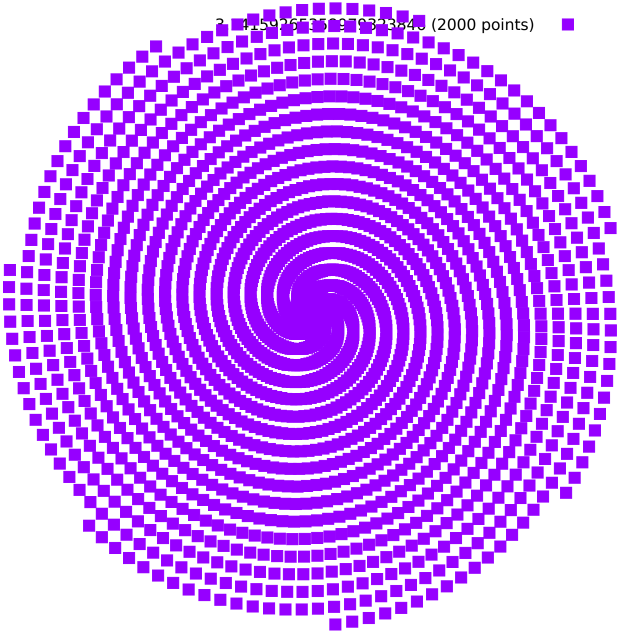
You are invited to write a fancy interactive JavaScript/canvas demo. Use the following simple formulas for the coordinates of the n'th point, where φ is the given angle to use ($\varphi = 1/4$ meaning 90 degrees).

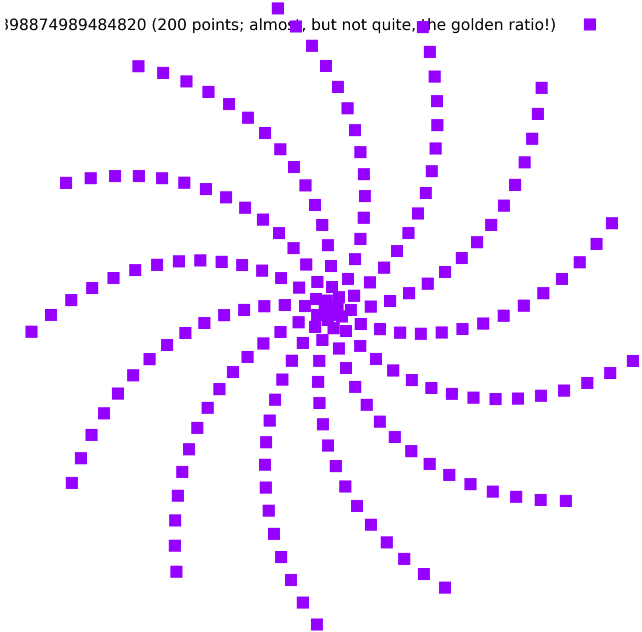
$$x = n \cdot \cos(2\pi\varphi \cdot n)$$
$$y = n \cdot \sin(2\pi\varphi \cdot n)$$

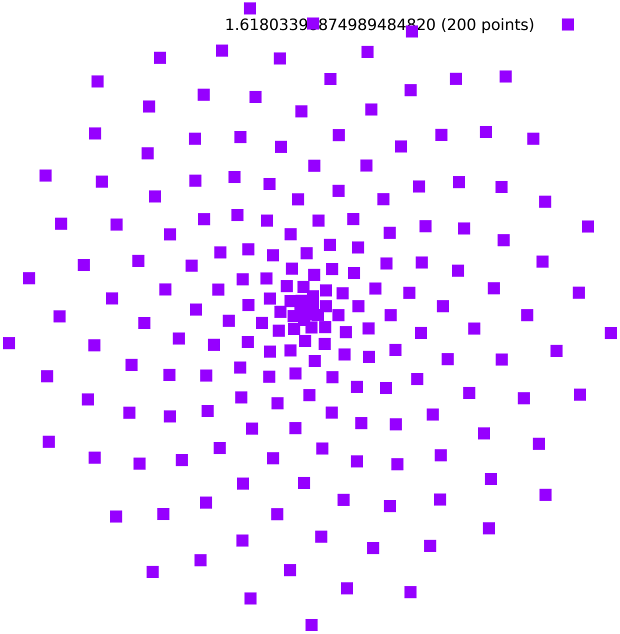

0.25 (200 points)

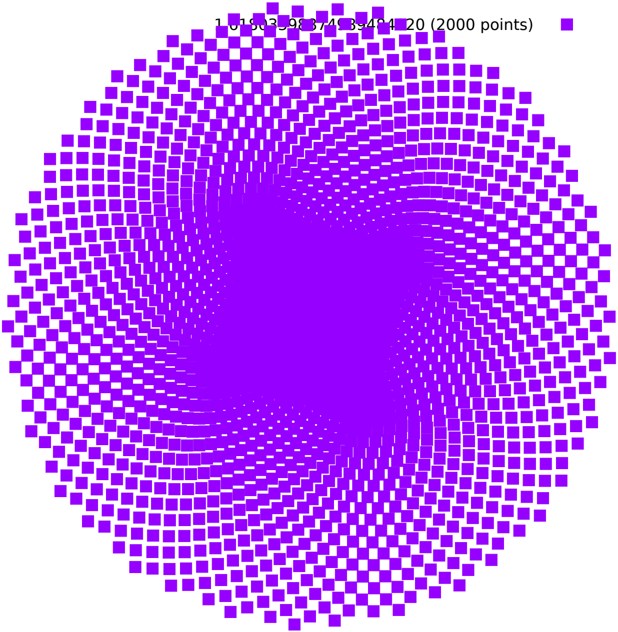

0.125 (200 points)


3.14159265358979323846 (200 points)


(2000 points)


398874989484820 (200 points; almost, but not quite, the golden ratio!)


1.61803398874989484820 (200 points)



20 (2000 points)

Why the Fibonacci numbers?

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

1. $1 \quad = \quad 1/1$
2. $[1; 1] \quad = \quad 2/1$
3. $[1; 1, 1]$

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4. $[1; 1, 1, 1] = 5/3$
5. $[1; 1, 1, 1, 1] = 8/5$
6. $[1; 1, 1, 1, 1, 1] = 13/8$
7. $[1; 1, 1, 1, 1, 1, 1] = 21/13$
8. $[1; 1, 1, 1, 1, 1, 1, 1] = 34/21$
9. $[1; 1, 1, 1, 1, 1, 1, 1, 1] = 55/34$

Using a fraction $\frac{a}{b}$ of the full circle as rotation angle (given in lowest terms) yields precisely b spirals. The animation at

`http://rawgit.com/iblech/number5/master/drehwinkel-0_30275229357799`

shows a zoom when using 33/109 as rotation angle. Its continued fraction expansion is

$$\frac{33}{109} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3}}}}$$

with truncations

$$\frac{1}{3}, \qquad \cfrac{1}{3 + \cfrac{1}{3}} = \frac{3}{10}, \qquad \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3}}} = \frac{10}{33}.$$

Therefore you first see three, then ten, then 33, and finally 109 spirals.

The pineapple from SpongeBob SquarePants

By Vi Hart, recreational mathemusician.

Watch *Open Letter to Nickelodeon, Re: SpongeBob's Pineapple under the Sea* by Vi Hart on YouTube: https://www.youtube.com/watch?v=gBxeju8dMho

Check out an exercise sheet for more fun:

http://rawgit.com/iblech/number5/master/pizzaseminar-en.pdf
http://rawgit.com/iblech/number5/master/pizzaseminar-de.pdf

Exercise 12 explains the relation between the golden ratio and the number 5.

Image sources

https://upload.wikimedia.org/wikipedia/commons/9/99/Vi_Hart.jpg
http://joachim-reichel.org/software/fraktal/mandelbrot_large.png
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