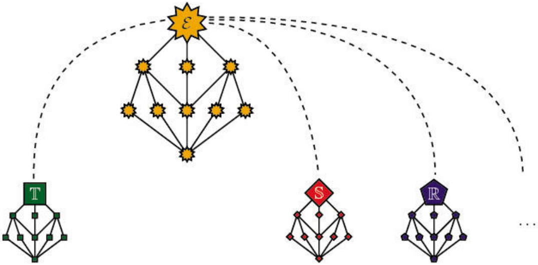

On the mystery of generic models

– an invitation –

Seminar at the
Dipartimento di Matematica (Tullio Levi-Civita)
Università degli Studi di Padova

June 25th, 2020

Ingo Blechschmidt

The generic ring

Let R be a ring.

$\mathbb{Z}$ $\mathbb{F}_2$ $\mathbb{Q}[X]$ $\mathbb{R}$ $\mathcal{O}_X$

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Thm. For any* property P of rings, the following are equivalent:

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2. Every* ring has property P.
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Example A. For any $x, y, z \in \mathbb{A}$, $x + (y + z) = (x + y) + z$.

Example B. It is **not the case** that $1 + 1 = 0$ in $\mathbb{A}$.

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Example C (Anders Kock). The generic ring is a **field**:

$$\forall x \in \mathbb{A}. \big((x = 0 \Rightarrow 1 = 0) \Rightarrow (\exists y \in \mathbb{A}.\, xy = 1)\big).$$

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Hence: When verifying a coherent sequent for all rings, can without loss of generality assume the field condition.

A selection of noncoherent sequents

The **generic object** $\mathbb{M}$ validates:

1. $\forall x, y \in \mathbb{M}.\, \neg\neg(x = y)$.
2. $\forall x_1, \ldots, x_n \in \mathbb{M}.\, \neg\forall y \in \mathbb{M}.\, y = x_1 \vee \cdots \vee y = x_n$.

The **generic ring** $\mathbb{A}$ validates:

1. $\forall x \in \mathbb{A}.\, (x = 0 \Rightarrow 1 = 0) \Rightarrow (\exists y \in \mathbb{A}.\, xy = 1)$.
2. $\forall x \in \mathbb{A}.\, \neg\neg(x = 0)$.

The **generic local ring** $\mathbb{A}'$ validates:

1. $\forall x \in \mathbb{A}'.\, (x = 0 \Rightarrow 1 = 0) \Rightarrow (\exists y \in \mathbb{A}'.\, xy = 1)$.
2. $\neg\forall x \in \mathbb{A}'.\, \neg\neg(x = 0)$.
3. $\forall f \in \mathbb{A}'[X]_{\text{degree}>0}.\, \neg\neg\exists x \in \mathbb{A}'.\, f(x) = 0$.

An application in commutative algebra

Let A be a reduced ring ($x^n = 0 \Rightarrow x = 0$). Let $\mathbb{p}$ be the **generic prime ideal*** of A. Then $A_\mathbb{p} := A[\mathbb{p}^{-1}]$ validates:

$A_\mathbb{p}$ is a **field**: $\forall x \in A_\mathbb{p}. (\neg(\exists y \in A_\mathbb{p}. xy = 1) \Rightarrow x = 0)$.
$A_\mathbb{p}$ has **¬¬-stable equality**: $\forall x, y \in A_\mathbb{p}. \neg\neg(x = y) \Rightarrow x = y$.
$A_\mathbb{p}$ is **anonymously Noetherian**.

This observation unlocks a short and conceptual proof of Grothendieck's **generic freeness lemma** in algebraic geometry.

Thm. (baby freeness) Let M be an A-module. Then 1 implies 3.

1 M is finitely generated ($\Longleftrightarrow M_\mathbb{p}$ is finitely generated)
2 M is locally free ($\Longleftrightarrow M_\mathbb{p}$ is free)
3 M is locally free *on a dense open* ($\Longleftrightarrow M_\mathbb{p}$ is *not not* free)

Proof. Elementary linear algebra over $A_\mathbb{p}$. □

A systematic source

> A. Kock has pointed out [**5** (ii)] that the generic local ring satisfies the nongeometric sentence
>
> $$\forall x_1 \dots \forall x_n . (\neg (\bigwedge_i (x_i = 0)) \rightarrow \bigvee_i (\exists y . x_i y = 1))$$
>
> which in classical logic defines a field! The problem of characterising all the non-geometric properties of a generic model appears to be difficult. If the generic model of a geometric theory T satisfies a sentence α then any geometric consequence of $T+(\alpha)$ has to be a consequence of T. We might call α T-redundant. Does the generic T-model satisfy all T-redundant sentences?

Gavin Wraith. *Some recent developments in topos theory.*
In: Proc. of the ICM (Helsinki, 1978).

Thm. (Nullstellensatz): The generic $\mathbb{T}$-model U_T validates:
For any coherent sequent σ,

$$\sigma \text{ holds for } U_{\mathbb{T}} \iff \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves } \sigma .$$

Thm. (universality): The generic $\mathbb{T}$-model validates a first-order formula P if and only if P is intuitionistically deducible from the axioms of $\mathbb{T}$ and the Nullstellensatz.

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Places where we can do mathematics (among others):

1. Set (sets)
2. Eff (data types)
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Definition. An *arithmetic universe* is a category with finite limits ("$\times$"), stable finite disjoint coproducts ("$\amalg$"), stable effective quotients ("$X/\sim$") and parametrized list objects ("$\mathbb{N}$", "$\mathrm{List}(X)$").

Thm. Any statement which is provable in **predicative constructive mathematics** (no powersets, no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice) is true in any arithmetic universe.

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Further examples:

5. the **initial** arithmetic universe
6. the **classifying** arithmetic universe for the theory of rings