

– an invitation –

Exploring hypercomputation with the effective topos

IMJ-PRG, Paris
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- 1 Crash course on ordinal numbers
- 2 (Super) Turing machines
 - Basics on Turing machines
 - Basics on super Turing machines
 - The power of super Turing machines
 - Outlook on the larger theory
- 3 The effective topos
 - First steps in the effective topos
 - Curious size phenomena
 - Wrapping up

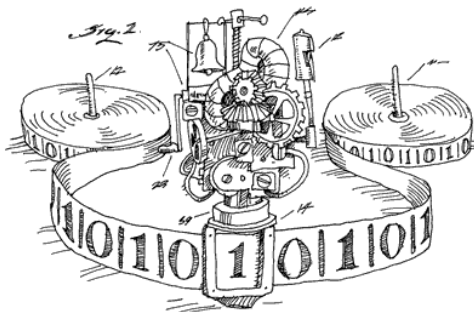
Part I

A crash course on ordinal numbers



Part II

(Super) Turing machines



Basics on Turing machines

- Turing machines are idealised computers operating on an **infinite tape** according to a **finite list** of rules.
- The concept is astoundingly robust.
- A subset of \mathbb{N} is **enumerable by a Turing machine** if and only if it's a Σ_1 -set.



Alan Turing
(* 1912, † 1954)



worth watching



Alison Bechdel
(* 1960)

Super Turing machines

With super Turing machines, the time axis is more interesting:

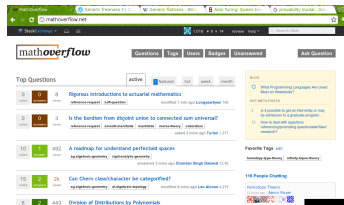
- normal: 0, 1, 2, ...
- super: 0, 1, 2, ..., ω , $\omega + 1$, ..., $\omega \cdot 2$, $\omega \cdot 2 + 1$,

On reaching a limit ordinal time step like ω or $\omega \cdot 2$,

- the machine is put into a designated state,
- the read/write head is moved to the start of the tape, and
- the tape is set to the “lim sup” of all its previous contents.



Joel David Hamkins



MathOverflow



Andy Lewis

A question for you

What's the behaviour of this super Turing machine?

In the start state and the limit state, check whether the current cell contains a “1”.

- If yes, then stop.
- If not, then flash that cell: set it to “1”, then reset it to “0”. Then unremittingly move the head rightwards.

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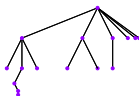
- If yes, then stop.
- If not, then flash that cell: set it to “1”, then reset it to “0”. Then unremittingly move the head rightwards.

This machine halts after time step ω^2 .

**Super Turing machines can break out of
(some kinds of) infinite loops.**

What can super Turing machines do?

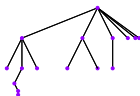
- Everything ordinary Turing machines can do.
- Verify number-theoretic statements.
- Decide whether a given ordinary Turing machine halts.
- Simulate super Turing machines.
- Decide Π_1^1 - and Σ_1^1 -statements:
 - “For every function $\mathbb{N} \rightarrow \mathbb{N}$ it holds that ...”
 - “There is a function $\mathbb{N} \rightarrow \mathbb{N}$ such that ...”



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But: Super Turing machines can't compute all functions and can't write every 0/1-sequence to the tape.



Fun facts

- Every super Turing machine either halts or gets caught in an unbreakable infinite loop after **countably many steps**.
- An ordinal number α is **clockable** iff there is a super Turing machine which halts precisely after time step α .
 - Speed-up Lemma: If $\alpha + n$ is clockable, then so is α .
 - Big Gaps Theorem
 - Many Gaps Theorem
 - Gapless Blocks Theorem
- **Lost Melody Theorem:** There are 0/1-sequences which a super Turing machine can recognise, but not write to the tape.

Part III

The effective topos



The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 After every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) \models 1" amounts to: There is a Turing machine which determines of any given number whether it is prime or not.

The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 After every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

“Eff(TM) \models 2” amounts to: There is a Turing machine which, given a number n , computes a prime larger than n .

The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 After every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	?	?
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) \models 3" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \rightarrow \mathbb{N}$, determines whether f has a zero or not.

The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 After every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	✓ (trivially)	?
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

“Eff(TM) \models 4” amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \rightarrow \mathbb{N}$, outputs a Turing machine computing f .

The effective topos

statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 After every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	?	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

“Eff(STM) \models 4” amounts to: There is a super Turing machine which, given a *super* Turing machine computing a map $f : \mathbb{N} \rightarrow \mathbb{N}$, outputs an (*ordinary*) Turing machine computing f .

The effective topos

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1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 After every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	✓ (if MP)	?
6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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6 Markov's principle holds.	✓ (trivially)	?	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

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6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	?
7 Countable choice holds.	✓	?	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

“Eff(TM) \models 6” amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \rightarrow \mathbb{N}$ and given the promise that it is *not not* the case that f has a zero, determines a zero of f .

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statement	in Set	in Eff(TM)	in Eff(STM)
1 Every number is prime or not prime.	✓ (trivially)	✓	✓
2 After every number there is a prime.	✓	✓	✓
3 Every map $\mathbb{N} \rightarrow \mathbb{N}$ has a zero or not.	✓ (trivially)	✗	✓
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5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7 Countable choice holds.	✓	?	?
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5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
7 Countable choice holds.	✓	✓ (always!)	?
8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	?	?

"Eff(TM) \models 7" amounts to: There is a Turing machine which, given a Turing machine computing for every $x \in \mathbb{N}$ some $y \in A$ together with a witness of $\varphi(x, y)$, outputs a Turing machine computing a suitable choice function $\mathbb{N} \rightarrow A$.

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4 Every map $\mathbb{N} \rightarrow \mathbb{N}$ is computable.	✗	✓ (trivially)	✗
5 Every map $\mathbb{R} \rightarrow \mathbb{R}$ is continuous.	✗	✓ (if MP)	✗
6 Markov's principle holds.	✓ (trivially)	✓ (if MP)	✓ (if MP)
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8 There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$.	✗	✗	✓

Curious size phenomena

$\text{Eff}(\text{STM}) \models \text{“There exists an injection } \mathbb{N}^{\mathbb{N}} \hookrightarrow \mathbb{N}.”$

means:

There is a super Turing machine which inputs the source of a super Turing machine A computing a function $\mathbb{N} \rightarrow \mathbb{N}$ and outputs a number $n(A)$ such that $n(A) = n(B)$ if and only if A and B compute the same function.

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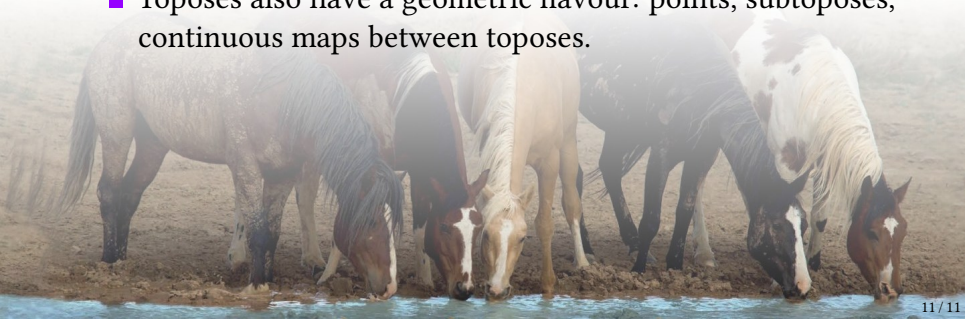
There is a super Turing machine which inputs the source of a super Turing machine A computing a function $\mathbb{N} \rightarrow \mathbb{N}$ and outputs a number $n(A)$ such that $n(A) = n(B)$ if and only if A and B compute the same function.

This statement is witnessed by following super Turing machine:

Read the source of a super Turing machine A from the tape. Simulate all super Turing machines in a dovetailing fashion. As soon a machine is found which has the same input/output behaviour as A , output the number of this machine and halt.

Wrapping up

- Effective toposes are a good vehicle for studying the nature of computation.
- Effective toposes build links between constructive mathematics and programming.
- Toposes allow for curious dream axioms.
- Toposes also have a geometric flavour: points, subtoposes, continuous maps between toposes.



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There is more to mathematics than the standard topos.