– an invitation –

A **modal logical multiverse** for commutative algebra and combinatorics

Séminaire Général de Logique
IMJ-PRG

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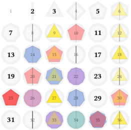
Ingo Blechschmidt
University of Augsburg

Three proofs



Thm. For every n, there is a prime larger than n.

Proof. Any prime factor of $n! + 1$ will do. $\square$

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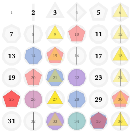

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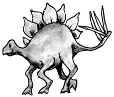

Thm. Every infinite sequence $\alpha : \mathbb{N} \to \mathbb{N}$ is *good* in that there are numbers $i < j$ such that $\alpha(i) \leq \alpha(j)$.

Proof. There is a minimal value $\alpha(i)$. Set $j := i+1$. □

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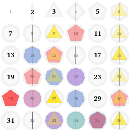

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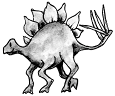

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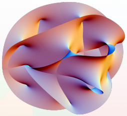

Thm. Let M be a surjective matrix with more rows than columns over a ring A. Then $1 = 0$ in A.

Proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is a field, this is a contradiction to basic linear algebra.

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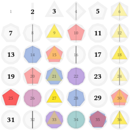

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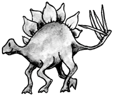

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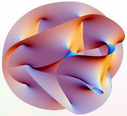


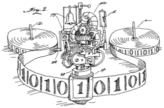
Thm. Let M be a surjective matrix with more rows than columns over a ring A. Then $1 = 0$ in A.

Proof. Write $M = \begin{pmatrix} x \\ y \end{pmatrix}$. By surjectivity, have u, v with

$$u \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence $1 = (vy)(ux) = (uy)(vx) = 0$. $\square$

Three take-aways



Abstract proofs should be blueprints for concrete ones.

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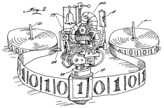

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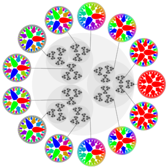

For every inhabited set X, there is a **flavor of parametrized mathematics** which is home to a gadget called **generic surjection** $\mathbb{N} \twoheadrightarrow X$.

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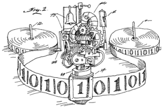

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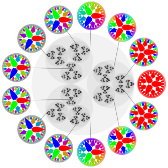

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There is a **rich modal multiverse** of flavors of parametrized mathematics.

A brief timeline

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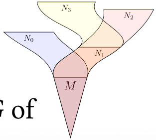
- 2016 · · · · · Oldenziel proposes to study the modal multiverse of parametrized mathematics.

The set-theoretic multiverse

Def. A **model of set theory** is a (perhaps class-sized) structure $(M, \in)$ satisfying axioms such as those of ZFC.

Examples.



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- L, Gödel's constructible universe
- $V[G]$, a forcing extension containing a generic filter G of some poset of forcing conditions
- Henkin/term models from consistency of (extensions of) ZFC

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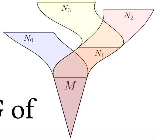

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$\Box\varphi$ iff φ holds in **all extensions** of the current universe.

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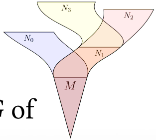

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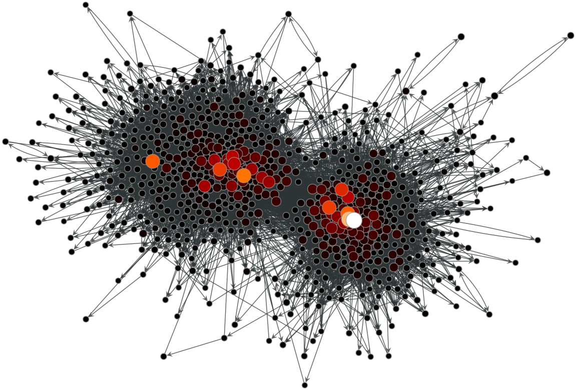
NB. In the multiverse of extensions of a given field K,

- "there is a square root of -1" is a button and
- "every polynomial splits into linear factors" is a switch.

Parametrized mathematics



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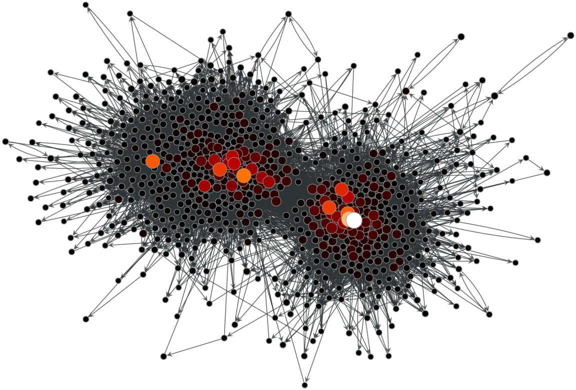

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“Let X be a topological space and let $A : X \to M_n^{\mathrm{sym}}(\mathbb{R})$ be a continuous map to the space of symmetric $(n \times n)$-matrices. Then there is an open covering $\bigcup_{i \in I} U_i$ of X such that or all indices $i \in I$, there is a continuous map $v : U_i \to \mathbb{R}^n$ such that for all $x \in U_i$, the vector $v(x)$ is an eigenvector of $A(x)$.”

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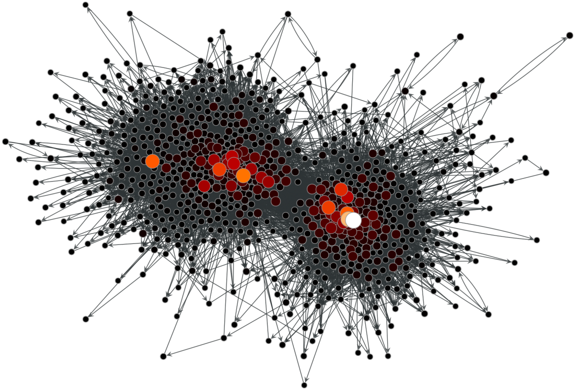

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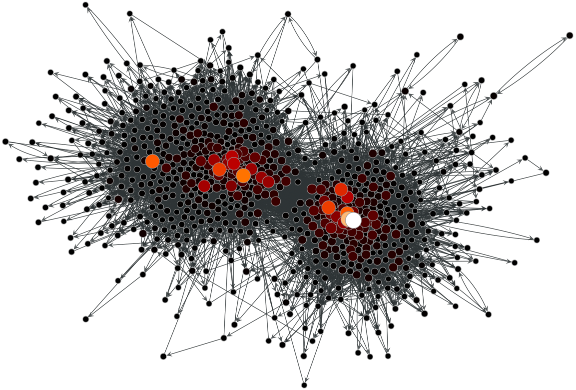

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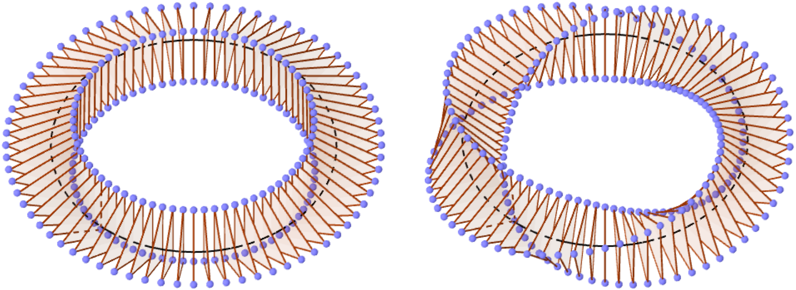

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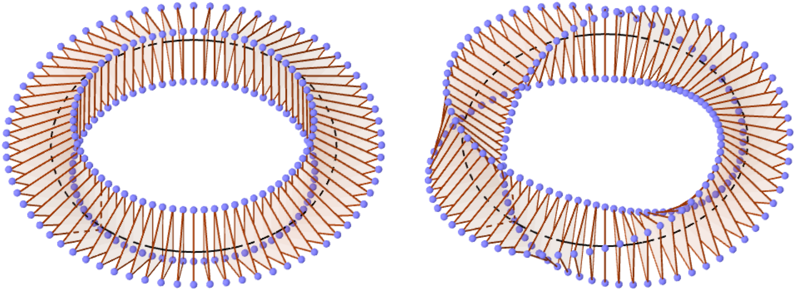

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"Let M be a finitely generated module over an arbitrary commutative ring A. Then there is a partition $1 = f_1 + \cdots + f_n \in A$ of unity such that, for each index i, the localized module $M[f_i^{-1}]$ is finite free over $A[f_i^{-1}]$."

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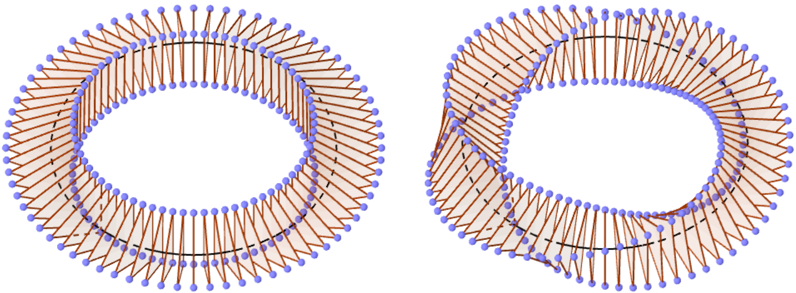

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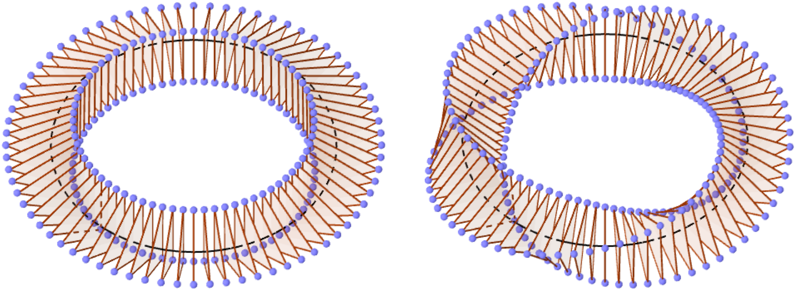

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Then M is **not not** finite free.” ✓

global “Let M be a finitely generated module over a ring A.
Then $M^{\sim}$ is finite locally free **on a dense open**.” ✓

“Let M be a finitely generated module over an arbitrary commutative ring A. **If $f = 0$ is the only element of A** such that $M[f^{-1}]$ is finite free over $A[f^{-1}]$, **then $1 = 0$ in A**.”

Parametrized mathematics

local “Let R be a ring. Let $n \geq 0$ be an integer. We have

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \textit{(omitted)}$$

as R-modules.” ✓

global “Let S be a scheme. Let $n \geq 1$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$. For the structure morphism $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S$, we have

$$R^q\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) = \textit{(omitted)}$$

as sheaves of $\mathcal{O}_S$-modules.” ✓

Sites and toposes

For every **site** $\mathcal{C}$, the **sheaves** on $\mathcal{C}$ form a **topos**. *Notable sites:*

1. The **site of opens** of a topological space
2. The **Zariski site** of a ring
3. The **site of finite approximations** of surjections $\mathbb{N} \twoheadrightarrow X$
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Def. A statement φ holds …

- **everywhere** ($\Box\varphi$) iff it holds in every (Grothendieck) topos (over the current base topos).
- **somewhere** ($\Diamond\varphi$) iff it holds in some positive topos.
- **proximally** ($\diamond\!\!\!\Diamond\varphi$) iff it holds in some positive ouvert topos.

The multiverse of parametrized mathematics

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1. *Somewhere*, the law of excluded middle holds. *In fact, we even have:* It is *everywhere* the case that the law of excluded middle holds *somewhere.*
2. Assuming Zorn's lemma, it is everywhere the case that it the axiom of choice holds somewhere. [Barr]
3. If a geometric implication holds *somewhere*, then it holds already here.
4. If a first-order statement holds *proximally*, then it holds already here.
5. For every inhabited set X, *proximally* there is a surjection $\mathbb{N} \twoheadrightarrow X$.
6. Every ring *proximally* has a maximal ideal.

```
module _ (A : Set) where

open import Data.List
open import Data.List.Membership.Propositional
open import Data.Product

data Eventually (P : List A → Set) : List A → Set where
  now
    : {σ : List A}
    → P σ
    → Eventually P σ
  later
    : {σ : List A} {a : A}
    → ((τ : List A) → a ∈ (σ ++ τ) → Eventually P (σ ++ τ))
    → Eventually P σ

State : (List A → Set) → (List A → Set)
State P σ = ((τ : List A) → Σ[ υ ∈ List A ] P (σ ++ τ ++ υ))
U:**-  Countable.agda   All L1    <N>    (Agda:Checked +5 Undo-Tree)

U:%*-  *All Done*   All L1    <M>    (AgdaInfo Undo-Tree)
```

Agda formalization available.