How not to constructivize cohomology

– interruptions welcome at any point –

Ingo Blechschmidt
University of Verona

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Flabby sets

Let M be a set. A subset $K \subseteq M$ is …

- a **subterminal** iff $\forall x, y \in K.\, x = y$.
- a **subsingleton** iff $\exists a \in M.\, \forall x \in K.\, x = a$, that is,
  iff $K \subseteq \{a\}$ for some $a \in M$.

Trivially, any subsingleton is a subterminal.

Definition. The set M is **flabby** iff any subterminal is a subsingleton.

Any flabby set is inhabited.

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Proposition. Any set embeds into a flabby set.

Proof. We have $M \hookrightarrow P(M)$, and $P(M)$ is flabby: Let $K \subseteq P(M)$ be a subterminal. Then $K \subseteq \{\bigcup K\}$, for if $A \in K$, then $K = \{A\}$ and hence $A \in \{\bigcup K\} = \{A\}$.

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Open question. Does any module embed into a flabby module?

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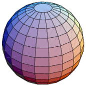
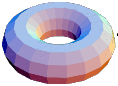
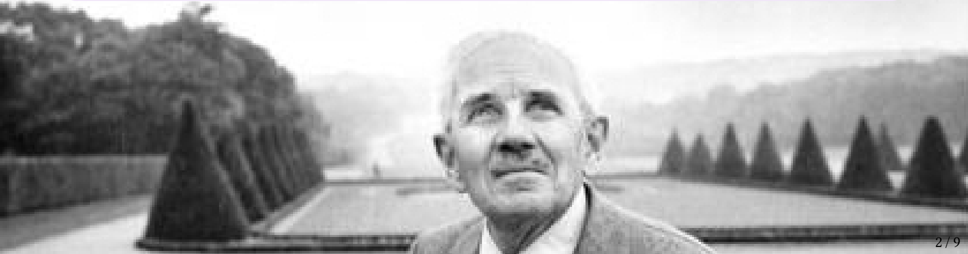
Singular cohomology

Is  homeomorphic to ? No:

$H^0_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$ $H^1_{\text{sing}}($$, \mathbb{Z}) \cong 0$ $H^2_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$

$H^0_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$ $H^1_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ $H^2_{\text{sing}}($$, \mathbb{Z}) \cong \mathbb{Z}$

Given $f : X \to B$, can we compute the cohomology of X if we understand the cohomology of B and the cohomology of the fibers of f?



Sheaf cohomology

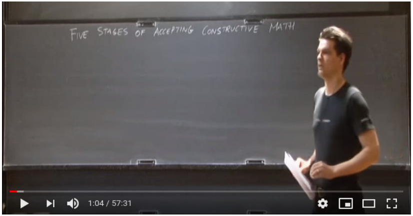
Let E be a sheaf of modules over a space X. Let Γ be the global sections functor. Choose an injective resolution $0 \to E \to I^0 \to I^1 \to \cdots$. Then the ***n*-th cohomology of *E* is**

$$H^n(X, E) := n\text{-th cohomology of } (0 \to \Gamma I^0 \to \Gamma I^1 \to \cdots)$$
$$= \ker(\Gamma I^n \to \Gamma I^{n+1}) \,/\, \mathrm{im}(\Gamma I^{n-1} \to \Gamma I^n).$$

- The modules $H^n(X, E)$ are important invariants.
  [$\chi(X, \mathcal{O}_X) = 1 - \mathrm{genus}_X$, $(C \cdot C') = \chi(\mathcal{O}_C \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}'_C), \ldots$]
- Let A be an abelian group. Let X be semi-locally contractible.
  Then $H^n(X, \underline{A}) = H^n_{\mathrm{sing}}(X, A)$ [Sella 2016].
- Let $f : X \to B$ be continuous. Then there is a spectral sequence
  $H^i(B, R^j f_*(E)) \Longrightarrow H^{i+j}(X, E)$.

Constructive mathematics

mathematics without $\varphi \vee \neg\varphi$, $\neg\neg\varphi \Rightarrow \varphi$, axiom of choice



Andrej Bauer at an IAS talk

Axiomatic freedom

"Every map $\mathbb{N} \to \mathbb{N}$ is computable."
"Every map $\mathbb{R} \to \mathbb{R}$ is continuous."
"Every map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is polynomial."
"Heyting Arithmetic has exactly one model."
"The subsets of $\{\heartsuit\}$ form a proper class."
"There is an injection $\mathbb{R} \to \mathbb{N}$."
$\vdots$

Applications

program extraction
synthetic differential geometry
synthetic algebraic geometry
synthetic domain theory
new reduction techniques in algebra
Bohr topos for quantum mechanics
$\vdots$

Relativization by internalization

Let X be a space. The **internal language** of the topos $\mathrm{Sh}(X)$ allows us to reason about sheaves on X in **naive element-based terms**.

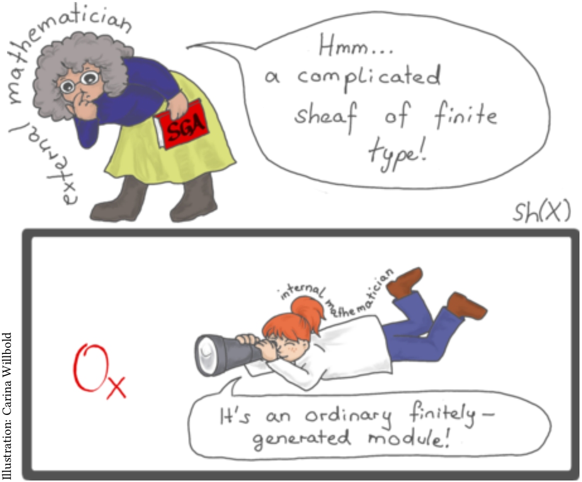


Illustration: Carina Willbold

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externally	internally to $\mathrm{Sh}(X)$
sheaf	set/type
morphism of sheaves	map between sets
sheaf of cont. real-valued functions	set of Dedekind reals
over-locale $f : Y \to X$	locale $I(Y)$
sheaf over Y	sheaf over $I(Y)$
higher direct image $R^n f_* E$	?? sheaf cohomology $H^n(I(Y), E)$

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Every finite type sheaf of modules is finite locally free *on a dense open*.	Every finitely generated vector space is *not not* finite free.
In continuous families of continuous functions with opposite signs, zeros can locally be picked continuously.	The intermediate value theorem holds.
Grothendieck's generic freeness lemma holds.	(Some trivial observation about modules over fields.)

Internalizing higher direct images

A set M is **injective** iff for any injection $A \to B$, any map $A \to M$ extends to a map on B.

- "A set is injective iff it's inhabited" is a **constructive taboo**.
- Constructively, there are still **enough injective sets**.
- Any injective set is flabby.

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A module M is **injective** iff for any linear injection $A \to B$, any linear map $A \to M$ extends to a linear map on B.

- It's consistent with **ZF** that there are no injective modules [Blass 1979].
- The existence of enough injective modules is **constructively neutral**.

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A sheaf of modules M is **injective** iff for any linear monomorphism $A \rightarrowtail B$, any linear morphism $A \to M$ extends to a linear morphism on B.

- Assuming choice, there are enough injectives over any site.
- Assuming Zorn's lemma, a sheaf of modules over a locale X is injective iff, from the internal point of view of $\mathrm{Sh}(X)$, it is an injective module.

Flabby resolutions

A sheaf E on a space X is **flabby** iff any local section $s \in E(U)$ on an open U extends to a global section $\overline{s} \in E(X)$: $\overline{s}|_U = s$.

- Assuming Zorn's lemma:
  A sheaf is flabby iff, from the internal point of view, it's a flabby set.
- Assuming the law of excluded middle:
  Any sheaf of modules over a topological space embeds into a flabby sheaf of modules.
- Assuming Zorn's lemma, flabby sheaves of modules are **acyclic for the global sections functor**. Hence, assuming ??, sheaf cohomology and higher direct images can be computed using **flabby resolutions**.

Flabbiness as an organizing principle

Proposition. Let M be a sheaf of modules over a locale X. Then M is injective iff it is injective from the point of view of $\mathrm{Sh}(X)$.

Proof. (Only "$\Leftarrow$".) Let $i : A \to B$ be a linear monomorphism. Let $f : A \to M$ be a linear morphism. Then verify, internally, that the set $E := \{\bar{f} : B \to M \mid \bar{f} \circ i = f\}$ is flabby.

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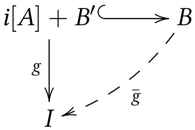
Let $K \subseteq E$ be a subterminal. We consider the injectivity diagram



where $B' := \{t \in B \mid t = 0 \text{ or } K \text{ is inhabited}\} \subseteq B$ and g is defined as follows: Let $s \in i[A] + B'$. Then $s = i(a) + t$ for some $a \in A$ and $t \in B'$. Since $t \in B'$, $t = 0$ or K is inhabited. If $t = 0$, we set $g(s) := f(a)$. If K is inhabited, we set $g(s) := f(a) + \bar{f}(s)$, where $\bar{f}$ is any element of K.

Since M is injective, there exists a dotted map $\bar{g} \in E$. We have $K \subseteq \{\bar{g}\}$. □

Flabbiness in the effective topos

A set M is **flabby** iff any subterminal $K \subseteq M$ is a subsingleton.

$\forall x, y \in K.\, x = y$

$\exists a \in M.\, K \subseteq \{a\}$

Proposition. Let X be an effective object in the effective topos. Then

"If X is flabby, any endomap on X has a fixed point."

from the point of view of the effective topos.

Proof (sketch). We have a procedure which computes for any subterminal $K \subseteq X$ an element a_K such that $K \subseteq \{a_K\}$. Let $f : X \to X$ be a map. Construct $K := \{f(a_K)\}$. Then $K \subseteq \{a_K\}$, so $f(a_K) = a_K$. □

Corollary. The only effective flabby module M is the zero module.

Proof. Let $x \in M$. Then $x + a = a$ for some $a \in M$; hence $x = 0$. □

Proposition. Assuming the law of excluded middle, any $\neg\neg$-separated module in the effective topos can be embedded into a flabby module.

Proof. We have $M \hookrightarrow \Delta\Gamma M$. □

State of affairs

☺

The existence of enough injective modules is constructively neutral.
Higher direct images can be understood as internal sheaf cohomology.

☹

Flabby sheaves can fail to be acyclic, constructively.
There is still no general constructive framework for sheaf cohomology.
Even though:

- Basic homological algebra is entirely constructive.
- There are algorithms for computing cohomology [Barakat, . . .].
- Čech methods work constructively, even in a synthetic context.