Without loss of generality, any reduced ring is a field

– an invitation –

REDCOM:
Reducing complexity in algebra, logic, combinatorics

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Transfinite methods in algebra?

Let A be a ring which is reduced: If $x^n = 0$, then $x = 0$.

Injective matrices

Theorem. Let M be an injective matrix with more columns than rows over A. Then $1 = 0$ in A.

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

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$$A_\mathfrak{p} = A[S^{-1}] = \{ \tfrac{x}{s} \mid x \in A, s \notin \mathfrak{p} \} \qquad \text{where } S = A \setminus \mathfrak{p}$$

$$A[f^{-1}] = A[S^{-1}] = \{ \tfrac{x}{f^n} \mid x \in A, n \geq 0 \} \qquad \text{where } S = \{f^0, f^1, \ldots\}$$

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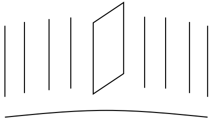
Grothendieck's generic freeness

Theorem. Let M be a finitely generated A-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is a free $A[f^{-1}]$-module, then $1 = 0$ in A.

Proof. See [Stacks Project]. □



A remarkable sheaf

Let A be a ring. Then there is a certain related 'ring' $A^\sim$ such that …

$A^\sim$ is close to A

1. $A^\sim$ inherits any property of A which is **localization-stable**.
2. A geometric sequent holds for $A^\sim$ iff* it holds for **all stalks** $A_\mathfrak{p}$.

$A^\sim$ is better than A

(Now assume A reduced.)

a. $A^\sim$ is a **field**: $\forall x : A^\sim.\ (\neg(\exists y : A^\sim.\ xy = 1) \Rightarrow x = 0)$.
b. $A^\sim$ has **¬¬-stable equality**: $\forall x, y : A^\sim.\ \neg\neg(x = y) \Rightarrow x = y$.
c. $A^\sim$ is **anonymously Noetherian**.

This sheaf can be exploited to give short, conceptual and constructive proofs.

The sheaf model as a local lens

sheaf model $A^\sim$	**ring A**
'$(\forall x : A^\sim .\, \varphi(x))$'	for all $g \in A$ and $x \in A[g^{-1}]$, '$\varphi(x)$ on $D(g)$'
'$\varphi \Rightarrow \psi$'	for all $g \in A$, 'φ on $D(g)$' implies 'ψ on $D(g)$'
'$\varphi \vee \psi$'	there is a partition $1 = g_1 + \cdots + g_m$ s. th. for each i, 'φ on $D(g_i)$' or 'ψ on $D(g_i)$'
'$\perp$'	$1 = 0$ in A

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On the field property

Let $x \in A$ be such that

'$x \in A^\sim$ is not invertible on $D(1)$',

that is

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for all $g \in A$, if '$x \in A^\sim$ is invertible on $D(g)$' then '$\bot$ on $D(g)$',

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for all $g \in A$, if x is invertible in $A[g^{-1}]$ then g is nilpotent.

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that is

$$\text{for all } g \in A, \text{ if } x \text{ is invertible in } A[g^{-1}] \text{ then } g \text{ is nilpotent.}$$

Then, considering $g := x$, it follows that $x = 0$ in A.

Revisiting the test cases

Injective matrices

Theorem. Let M be an injective matrix with more columns than rows over a ring A. Then $1 = 0$ in A.

Proof. Assume not. Then there is a minimal prime ideal $\mathfrak{p} \subseteq A$. The matrix is injective over the field $A_{\mathfrak{p}}$; contradiction to basic linear algebra. □

Proof. 'M is also injective as a matrix over $A^{\sim}$. This is a contradiction by basic intuitionistic linear algebra.' Thus '$\bot$'. Hence $1 = 0$ in A. □

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Proof. The claim amounts to '$M^\sim$ is **not not** free'. This statement follows from basic intuitionistic linear algebra over the field $A^\sim$. □

Forcing locality

Definition. A ring is **local** iff $1 \neq 0$ and $x + y = 1$ implies that x is invertible or y is invertible.

Examples: $k,\ k[[X]],\ \mathbb{C}\{z\},\ \mathbb{Z}_{(p)}$
Non-examples: $\mathbb{Z},\ k[X],\ \mathbb{Z}/(pq)$

Not every ring A is local. But always: '$A^\sim$ is local on $D(1)$'

Locally, any ring is local: Let $x + y = 1$ in a ring A. Then:

- The element x is invertible in $A[x^{-1}]$.
- The element y is invertible in $A[y^{-1}]$.
- $(D(x), D(y))$ *covers* $D(1)$.