1

Lösungen Blatt 9

Aufgabe 33:

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

i) Beh: $A \in O(3)$

Beweis: z.z. ${}^{T}\!A\,A = \mathbb{1}$

$$\begin{array}{cc} & \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix} \\ \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

ii) Beh: $S = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ und ${}^{T}SAS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$, $\alpha = \frac{\pi}{3}$

Beweis:

Nach Vorlesg hat A entweder EW 1 oder EW -1

$$\mathrm{Eig}(1) = \ker \frac{1}{3}\begin{pmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} = \ker \begin{pmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} = \ker \begin{pmatrix} -1 & -1 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

Ergänze zu einer Basis von $\mathbb{R}^3$ und orthonormalisiere:

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_2' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left\langle \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \Rightarrow v_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$w_3' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \left\langle \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} - \left\langle \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

② $\Rightarrow$ ONB: $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$

Setze $S = \left(v_1 \middle| v_2 \middle| v_3\right) \in O(3) \Rightarrow S^{-1} = {}^tS = \begin{pmatrix}{}^tv_1\\{}^tv_2\\{}^tv_3\end{pmatrix}$

und

$${}^tS\,A\,S = \frac{1}{3}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2 & -1 & 2\\ 2 & 2 & -1\\ -1 & 2 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ 0 & \frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\frac{\pi}{3} & -\sin\frac{\pi}{3}\\ 0 & \sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}$$

(3)

Aufgabe 34:

$$U = \frac{1}{4}\begin{pmatrix} \sqrt{2} + i(\sqrt{2}-2) & (\sqrt{2}+2) - i\sqrt{2} \\ -(\sqrt{2}+2) + i\sqrt{2} & \sqrt{2} + i(\sqrt{2}-2) \end{pmatrix}$$

i) Beh: U ist unitär

Beweis: z.z. $^t\bar{U} \cdot U = 1\!\!1$

$$^t\bar{U} = \frac{1}{4}\begin{pmatrix} \sqrt{2} - i(\sqrt{2}-2) & -(\sqrt{2}+2) - i\sqrt{2} \\ (\sqrt{2}+2) + i\sqrt{2} & \sqrt{2} - i(\sqrt{2}-2) \end{pmatrix}$$

$$^t\bar{U} \cdot U = \frac{1}{16}\begin{pmatrix} \sqrt{2} - i(\sqrt{2}-2) & -(\sqrt{2}+2) - i\sqrt{2} \\ (\sqrt{2}+2) + i\sqrt{2} & \sqrt{2} - i(\sqrt{2}-2) \end{pmatrix}\begin{pmatrix} \sqrt{2} + i(\sqrt{2}-2) & (\sqrt{2}+2) - i\sqrt{2} \\ -(\sqrt{2}+2) + i\sqrt{2} & \sqrt{2} + i(\sqrt{2}-2) \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 2 + (\sqrt{2}-2)^2 + (\sqrt{2}+2)^2 + 2 & 0 \\ 0 & (\sqrt{2}+2)^2 + 2 + 2 + (\sqrt{2}-2)^2 \end{pmatrix} = 1\!\!1$$

ii) Beh:

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \quad \text{und } T^{-1} U T = \begin{pmatrix} \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & 0 \\ 0 & -i \end{pmatrix}$$

Beweis:

$$\chi_U(x) = \det(U - x 1\!\!1) \overset{\det(r \cdot A) = r^n \cdot \det(A)}{=} \frac{1}{16} \det\begin{pmatrix} \sqrt{2} + i(\sqrt{2}-2) - 4x & (\sqrt{2}+2) - i\sqrt{2} \\ -(\sqrt{2}+2) + i\sqrt{2} & \sqrt{2} + i(\sqrt{2}-2) - 4x \end{pmatrix}$$

$$= x^2 - \tfrac{1}{2} x \left(\sqrt{2} + i(\sqrt{2}-2)\right) + \frac{\sqrt{2}}{2} = (x+i)\left(x - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$$

$\Rightarrow$ EW $\lambda_1 = -i$, $\lambda_2 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

(4) $Eig(-i) = \ker(A + i\mathbb{1})$

$$= \ker \begin{pmatrix} \sqrt{2} + i(\sqrt{2}-2) + 4i & -(\sqrt{2}+2) + i\sqrt{2} \\ (\sqrt{2}+2) - i\sqrt{2} & \sqrt{2} + i(\sqrt{2}-2) + 4i \end{pmatrix}$$

Es ist $v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \in Eig(-i)$

Ergänze zu einer ONB von $\mathbb{C}^2$

Suche w_2 mit $\begin{pmatrix} 1 \\ -i \end{pmatrix} \perp w$, d.h.

$$^t\overline{w_2} \cdot \overline{v_1} = 0 \Leftrightarrow (w_2^1\ w_2^2) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = 0 \Leftrightarrow w_2^1 + i w_2^2 = 0$$

$$\Leftrightarrow w_2^1 = -i \cdot w_2^2$$

Setze $w_2^1 = 1 \Rightarrow w_2^2 = i$

Also $v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ orthogonale Basis

Orthonormalisiere:

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\Rightarrow w_1, w_2$ ist ONB

Setze $T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, dann ist $^t\overline{T} = T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix}$

$$^t\overline{T} \cdot A \cdot T = \begin{pmatrix} \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & 0 \\ 0 & -i \end{pmatrix}$$