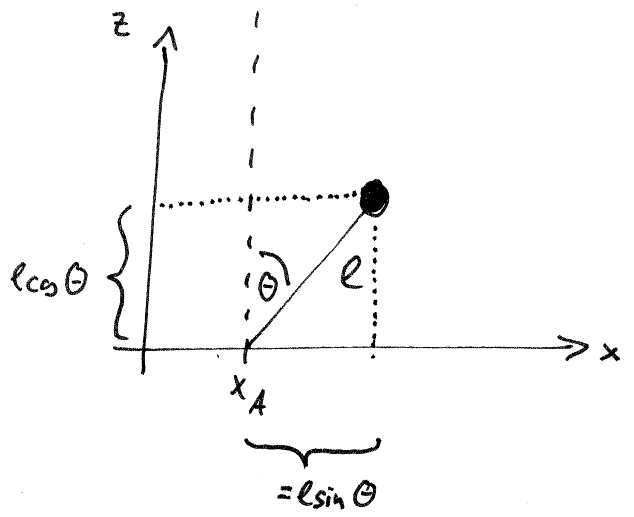
2012F / Umgekehrtes Pendel



$$x = x_A + \ell\sin\Theta$$

$$z = +\ell\cos\Theta$$

a) $\dot{x} = \dot{x}_A + \ell\cos\Theta\,\dot{\Theta}$, $\quad \dot{z} = -\ell\sin\Theta\,\dot{\Theta}$, $\quad \xi = \frac{x_A}{\ell}$, $\dot{\xi} = \frac{\dot{x}_A}{\ell}$, $\lambda = \sqrt{\frac{g}{\ell}}$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) = \frac{m}{2}\left(\dot{x}_A^2 + 2\ell\dot{x}_A\dot{\Theta}\cos\Theta + \ell^2\cos^2\Theta\,\dot{\Theta}^2 + \ell^2\sin^2\Theta\,\dot{\Theta}^2\right)$$

$$= \frac{m}{2}\left(\dot{x}_A^2 + 2\ell\cos\Theta\,\dot{x}_A\dot{\Theta} + \ell^2\dot{\Theta}^2\right) = \frac{m\ell^2}{2}\left(\frac{\dot{x}_A^2}{\ell^2} + 2\cos\Theta\,\frac{\dot{x}_A}{\ell}\dot{\Theta} + \dot{\Theta}^2\right)$$

$$= \frac{m\ell^2}{2}\left(\dot{\Theta}^2 + 2\dot{\xi}\dot{\Theta}\cos\Theta + \dot{\xi}^2\right)$$

$$V = mgz = mg\ell\cos\Theta$$

Lagrange-Funktion:

$$\mathcal{L} = T - V = \frac{m\ell^2}{2}\left(\dot{\Theta}^2 + 2\dot{\xi}\dot{\Theta}\cos\Theta + \dot{\xi}^2\right) - mg\ell\cos\Theta$$

Bewegungsgleichungen:

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\Theta}} - \frac{\partial\mathcal{L}}{\partial\Theta} = \frac{d}{dt}\left(m\ell^2\dot{\Theta} + m\ell^2\dot{\xi}\cos\Theta\right) - \left(+mg\ell\sin\Theta - m\ell^2\dot{\xi}\dot{\Theta}\sin\Theta\right)$$

$$= m\ell^2\ddot{\Theta} + m\ell^2\ddot{\xi}\cos\Theta \underline{- m\ell^2\dot{\xi}\dot{\Theta}\sin\Theta} - mg\ell\sin\Theta \underline{+ m\ell^2\dot{\xi}\dot{\Theta}\sin\Theta}$$

$$= m\ell^2\ddot{\Theta} + m\ell^2\ddot{\xi}\cos\Theta - mg\ell\sin\Theta$$

$$\Leftrightarrow \quad \ddot{\Theta} + \ddot{\xi}\cos\Theta - \lambda^2\sin\Theta = 0$$

b) $|\Theta| \ll 1, \quad \sin\Theta \approx \Theta, \quad \cos\Theta \approx 1$

$$\Rightarrow \quad \ddot{\Theta} + \ddot{\xi} - \lambda^2\Theta = 0$$

$$\ddot{\Theta} - \lambda^2\Theta = -\ddot{\xi}$$

c) Ansatz: $\Theta(t) = A e^{\mu t}$

$$A\mu^2 e^{\mu t} - \lambda^2 A e^{\mu t} = 0$$

$$\mu^2 = \lambda^2$$

$$\mu = \pm\lambda \quad \Rightarrow \quad \Theta^{(0)}(t) = \alpha e^{\lambda t} + \beta e^{-\lambda t}$$

2 Integrationskonstanten, da 2 AB, da DGl 2. Ordnung.

d) $\xi(t) = \xi_0 \sin\omega t, \qquad \ddot{\xi}(t) = -\xi_0\omega^2 \sin\omega t$

$$\Theta^{(1)}(t) = \Theta_0 \sin\omega t$$

Einsetzen von $\xi, \Theta^{(1)}$ in DGl $\ddot{\Theta} - \lambda^2\Theta = -\ddot{\xi}$:

$$-\Theta_0\omega^2 \sin\omega t - \lambda^2\Theta_0 \sin\omega t = \xi_0\omega^2 \sin\omega t$$

$$\Theta_0 = -\frac{\xi_0\omega^2}{\omega^2 + \lambda^2}$$

e) $$\Theta(t) = \alpha e^{\lambda t} + \beta e^{-\lambda t} + \Theta_0 \sin\omega t$$

Lösung beschränkt $\Leftrightarrow |\Theta(t \to \infty)| < \infty$

Da $\lambda = \sqrt{\frac{g}{\ell}} > 0$, muss gelten: $\alpha = 0$ (sonst ist die Lsg nicht beschränkt)

$$\Rightarrow \Theta(t) = \beta e^{-\lambda t} + \Theta_0 \sin\omega t$$

$\Theta(0) = \beta$

$\dot{\Theta}(0) = -\lambda\beta - \Theta_0\omega = -\lambda\Theta(0) - \Theta_0\omega$